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Gluon polarization tensor and dispersion relation in a weakly magnetized medium

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Abstract We study the polarization and dispersion properties of gluons moving within a weakly magnetized background at one-loop order. To this end, we show two alternative derivations of the charged fermion propagator in the weak field expansion and use this expression to compute the lowest order magnetic field correction to the gluon polarization tensor. We explicitly show that, in spite of its cumbersome appearance, the gluon polarization tensor is transverse as required by gauge invariance. We also show that none of the three polarization modes develops a magnetic mass and that gluons propagate along the light cone, non withstanding that Lorentz invariance is lost due to the presence of the magnetic field. We also study the strength of the polarization modes for real gluons. We conclude that the lowest order approximation to the gluon polarization and dispersion properties is good as long as the field strength and gluon momentum are not larger than the loop fermion mass. When the fermion mass is the vacuum one, the applicability of these findings for phenomenological studies is rather limited. However, should temperature be accounted for and the fermion mass become the thermal one, conditions met during the plasma phase of a heavy-ion collision, these findings can be potentially very useful to describe gluon mediated processes in the presence of a magnetic field.

1 Introduction

In recent years, the properties of strongly interacting matter in the presence of magnetic fields have received a great deal of attention. Motivated by the possibility that magnetic fields of a large intensity–albeit short-lived– can be produced in peripheral heavy-ion collisions at high energies, efforts, from the experimental as well as from the theoretical points of view, have been devoted to identify modifications induced on the propagation properties of quarks, gluons, and even hadrons in a magnetized medium. For instance, attempts have been made to link the observation of the charge separation along the magnetic field direction [1] with the chiral magnetic effect [2] or the anomalous excess of the yield and v_2 of direct photons [3–8] with contributions from channels otherwise not present in the absence of a magnetic field [9–14]. More recently, measurements of a different global polarization of Λ and $\overline{\Lambda}$, as the collision energy decreases [15], have been associated with the magnetic field produced in the reaction [16]. Also, lattice QCD (LQCD) calculations [17–19] have shown that for temperatures above the chiral restoration temperature, the quark-antiquark condensate decreases and that this temperature itself also decreases as the field intensity increases. This is the so-called *inverse magnetic catalysis* phenomenon and the search for its origin has been also intensively studied [20–32]. Moreover, LQCD has also shown that the magnetic field-driven modifications of neutral and charged mesons are different [33], with the neutral pion mass decreasing as the field intensity increases. In the linear sigma model, a possible explanation is found in the behavior of the coupling constants as function of the field strength [34].

When the magnetic field is the largest of the energy (squared) scales, a commonly used description to include its effects is the *strong field approximation*, whereby the propagation of charged particles occurs only within the Lowest

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Landau Level (LLL). The rationale behind this approximation is that fluctuations, whose typical energy is smaller than the separation between Landau Levels (proportional to the squared root of the field intensity), cannot induce transitions. This approach has been implemented in a recent calculation of the thermal corrections to the Debye magnetic mass [35] and can be relevant for the very early stages of a preripheral heavy-ion collision.

However, there are many physical situations where the magnetic field may not be the largest of the energy scales. This is for example the case some time after the very early stages of a peripheral heavy-ion collision, since the field intensity is a fast decreasing function of time. In addition, for the time when the thermal phase sets in, the relevant quark masses are the thermal ones, This opens up the rather interesting window of energy scales whereby the energy associated with the magnetic field, $\sqrt{eB}$, can become smaller than the thermal masses $m_f \sim g_s T$. In this situation, temperature effects have to be added on top of the magnetic background and the phenomenology associated with the hierarchy of scales $\sqrt{eB} < m_f < T$ is potentially very rich. Thus, it becomes interesting to first study the magnetic effects for the situation where $\sqrt{eB} < m_f$ to eventually add thermal effects on top of the magnetic background. In these situations the approximation, where the magnetic field is a small energy scale, is better suited. This scenario is best described by considering the propagation of charged particles in the *weak field approximation*. In this context, a recent calculation for the dissociation of heavy quarkonia in a weak magnetic field has been presented in Ref. [36].

The expressions for the charged fermion and scalar propagators in the weak field approximation were first obtained in Refs. [37,38], respectively. These propagators can be used, in particular, to find the quantum corrections to the propagation properties of neutral particles in a magnetized medium. In this work we concentrate on the computation of the gluon polarization tensor in the presence of a magnetic field at one-loop level using the weak field approximation to describe the fermion propagator. As we will show, the computation is plagued with subtleties, most notably, the enforcement of gauge invariance encoded in the transversality properties of the polarization tensor. We show that a systematic treatment of the weak field approximation ensures that terms that do not respect gauge invariance, vanish. The result is checked against the one obtained from the expansion to lowest non-trivial order in the field intensity of the general expression, which has been recently obtained in Ref. [39], and we find a coincident result between both approaches, provided the field intensity and the gluon momentum are small compared to the quark mass. The method hereby presented is better suited than an expression valid to all orders in the magnetic field, to be extended at finite temperature and density, a relevant scenario, given the current and future explorations of the QCD phase diagram in heavy-ion experiments in the STAR-BES, FAIR, and NICA experiments.

The work is organized as follows: In order to better identify the approximations that lead to the result for the charged fermion propagator in the weak field approximation obtained in Ref. [37], in Sect. 2 we present two different weak-field expansion methods to derive such propagator. We show the explicit result of this expansion up to $\mathcal{O}(B^6)$. In Sect. 3 we obtain the expression for the gluon polarization tensor in the weak field approximation to second order in the field intensity. We compare with the expression obtained from the expansion to the same order obtained in Ref. [39] and verify the gauge invariance of the result. In Sect. 4 we analyse the dispersion relation for the three different propagating modes from the coefficients of their tensor structures. We show that, in spite of their cumbersome appearance, the gluon does not develop a magnetic mass in any mode and moves along the light cone. We also study the strength of the polarization modes for on-shell gluons. Finally, we summarize and conclude in Sect. 5 leaving for the appendix the explicit derivation of the charged fermion propagator in the weak field expansion using the second method mentioned above.

2 Fermion propagator in the weak field limit

In order to compute the gluon polarization tensor in the presence of a weak magnetic field, we first proceed to obtain the expression for the charged fermion propagator that appears in the one-loop correction to the gluon polarization tensor in this limit. This object has been first obtained in Ref. [37], Nevertheless, we hereby present two alternative methods to obtain it. For the first method, we make a direct expansion in powers of the field strength starting from the full fermion propagator using Schwinger's proper-time representation. For the second method, we start from the full expression of the fermion propagator written in terms of an expansion of Landau Levels. In order to set the stage for the calculation, we first describe the notation and conventions to be used throughout the calculations.

2.1 Notation

Before we show the derivation of the fermion propagator in the weak field limit, here we spell out the notation and conventions that we use in this work. The magnetic field is taken as pointing along the $\hat{\mathbf{z}}$-axis and having an intensity $B = |\mathbf{B}|$, namely, $\mathbf{B} = B\hat{\mathbf{z}}$. The field couples to fermions of mass m_f through their electric charge q_f. In addition, we use the following notation and conventions:

– *Metric tensor*

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1}$$

– *Parallel and transverse metric tensors*

$$g^{\mu\nu} = g_{\parallel}^{\mu\nu} + g_{\perp}^{\mu\nu}, \tag{2}$$

with

$$g_{\parallel}^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3}$$

and

$$g_{\perp}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

– *Four-vectors*

$$X^{\mu} = (X_{\parallel}^{\mu} + X_{\perp}^{\mu}),$$
$$X_{\mu} = (X_{\mu}^{\parallel} - X_{\mu}^{\perp}), \tag{5}$$

with

$$X_{\parallel}^{\mu} = (X_0, 0, 0, X_3), \quad X_{\mu}^{\parallel} = (X_0, 0, 0, -X_3),$$
$$X_{\perp}^{\mu} = (0, X_1, X_2, 0), \quad X_{\mu}^{\perp} = (0, X_1, X_2, 0). \tag{6}$$

– *Squared four-vector*

$$X^{\mu} X_{\mu} = X^2 = X_{\parallel}^2 - X_{\perp}^2$$
$$= (X_0^2 - X_3^2) - (X_1^2 + X_2^2). \tag{7}$$

2.2 Weak field approximation from Schwinger formalism

The most general expression for the charged fermion propagator in the presence of a constant and uniform magnetic field is

$$iS(x, x') = e^{i\Phi(x,x')} \int \frac{d^4 p}{(2\pi)^4} e^{i(x-x')p}\, iS(p), \tag{8}$$

where $\Phi(x, x')$ is the so-called Schwinger's phase factor and $iS(p)$ is the translationally invariant term. Since, we are interested in computing the one-loop correction to the gluon polarization tensor, the phase factor vanishes and thus we hereby do not account for it.

We express $iS(p)$ using the Schwinger proper-time representation

$$iS(p) = i \int_0^{\infty} d\tau e^{\tau\left(p_{\parallel}^2 - p_{\perp}^2 \frac{\tanh(\tau|q_f B|)}{\tau|q_f B|} - m_f^2\right)}$$
$$\times \Bigg\{ [m_f + \not{p}_{\parallel}][1 + i\gamma^1\gamma^2 \tanh(\tau|q_f B|)\mathrm{sign}(q_f B)]$$
$$- \frac{\not{p}_{\perp}}{\cosh(\tau|q_f B|)^2} \Bigg\}. \tag{9}$$

Since in this work $|q_f B|$ is the smallest energy (squared) scale, in order to find a suitable representation of the propagator in the weak field approximation, we now proceed to perform a Taylor's series expansion of Eq. (9) for $\tau|q_f B| \sim 0$ and then to integrate over the proper time parameter τ. It can be shown by using this straightforward expansion that the propagator, written here up to $\mathcal{O}(B^6)$, becomes

$$iS(p) \simeq i\frac{m_f + \not{p}}{p^2 - m_f^2} - |q_f B|\gamma^1\gamma^2 \frac{m_f + \not{p}_{\parallel}}{(p^2 - m_f^2)^2}\mathrm{sign}(q_f B)$$
$$-2i|q_f B|^2 \frac{(m_f^2 - p_{\parallel}^2)\not{p}_{\perp} + p_{\perp}^2(m_f + \not{p}_{\parallel})}{(p^2 - m_f^2)^4}$$
$$+2|q_f B|^3\gamma^1\gamma^2 \frac{(m_f + \not{p}_{\parallel})(p_{\parallel}^2 + 3p_{\perp}^2 - m_f^2)}{(p^2 - m_f^2)^5}\mathrm{sign}(q_f B)$$
$$-8i|q_f B|^4 \frac{(2p_{\parallel}^2 + 3p_{\perp}^2 - 2m_f^2)}{(p^2 - m_f^2)^7}$$
$$\times \left[(m_f^2 - p_{\parallel}^2)\not{p}_{\perp} + p_{\perp}^2(m_f + \not{p}_{\parallel})\right]$$
$$-8|q_f B|^5\gamma^1\gamma^2 \frac{(m_f + \not{p}_{\parallel})}{(p^2 - m_f^2)^8}\mathrm{sign}(q_f B)$$
$$\times \left[18p_{\perp}^2(p_{\parallel}^2 - m_f^2) + 2(m_f^2 - p_{\parallel}^2)^2 + 15p_{\perp}^4\right]$$
$$+16i|q_f B|^6 \frac{\left[(m_f^2 - p_{\parallel}^2)\not{p}_{\perp} + p_{\perp}^2(m_f + \not{p}_{\parallel})\right]}{(p^2 - m_f^2)^{10}}$$
$$\times \left[78p_{\perp}^2(p_{\parallel}^2 - m_f^2) + 17(m_f^2 - p_{\parallel}^2)^2 + 45p_{\perp}^4\right]. \tag{10}$$

We notice that Eq. (10) coincides with the weak field expansion performed in Ref. [37]. It may be surprising that a simple Taylor's series expansion, starting out from Schwinger's proper time representation, yields the same result than the one in Ref. [37], where the result was obtained using a much more complicated method. The computation also reveals that the dominant region for the integration over the proper-time parameter is the small τ region and that, in order to obtain analytical results, this region can be extended to cover the whole original domain $0 \le \tau \le \infty$.

2.3 Weak field approximation from the Landau Levels representation

To find an expression of the fermion propagator in a weak magnetic field, we can also start from the expression of the full propagator written as a sum over Landau Levels [40,41]

$$iS(p) = ie^{-p_\perp^2/|q_f B|}\sum_{n=0}^{\infty}(-1)^n \frac{\mathcal{D}_n(q_f B, p)}{p_\parallel^2 - m_f^2 - 2n\,|q_f B|}. \quad (11)$$

The factor $\mathcal{D}_n(q_f B, p)$ is defined as

$$\begin{aligned}\mathcal{D}_n(q_f B, p) &= 2(\not{p}_\parallel + m_f)\mathcal{O}^- L_n^0\left(\frac{2p_\perp^2}{|q_f B|}\right)\\ &-2(\not{p}_\parallel + m_f)\mathcal{O}^+ L_{n-1}^0\left(\frac{2p_\perp^2}{|q_f B|}\right)\\ &+4\not{p}_\perp L_{n-1}^1\left(\frac{2p_\perp^2}{|q_f B|}\right), \end{aligned} \quad (12)$$

where $L_n^m(x)$ are the generalized Laguerre polynomials and

$$\mathcal{O}^\pm \equiv \frac{1}{2}\left[1 \pm i\ \mathrm{sign}(q_f B)\gamma^1\gamma^2\right]. \quad (13)$$

The desired approximation has to take into account the contribution from all Landau Levels. If we assume that the magnetic field is the weakest energy scale, a formal series representation for the fermion propagator in powers of $|q_f B|$ can be computed. For this purpose, notice that the denominators of Eq. (11) admit a geometrical series expansion, namely,

$$\frac{1}{p_\parallel^2 - m_f^2 - 2n\,|q_f B|} = \frac{1}{p_\parallel^2 - m_f^2}\sum_{k=0}^{\infty}\left(\frac{2n\,|q_f B|}{p_\parallel^2 - m_f^2}\right)^k. \quad (14)$$

From the above, the summation over the Landau level index n becomes

$$\mathcal{S}_k = \sum_{n=0}^{\infty}(-1)^n n^k L_n^m(2\alpha), \quad (15)$$

where $m = 0$ or 1 and $\alpha = p_\perp^2/eB$. To carry out this kind of sum, we use the identities

$$e^{-\alpha}\sum_{n=0}^{\infty}(-1)^n e^{-2inv}L_n^0(2\alpha) = \frac{e^{iv}}{2\cos v}e^{-i\alpha\tan v}, \quad (16a)$$

$$e^{-\alpha}\sum_{n=0}^{\infty}(-1)^n e^{-2i(n+1)v}L_n^1(2\alpha) = \frac{1}{4\cos^2 v}e^{-i\alpha\tan v}, \quad (16b)$$

so that

$$e^{-\alpha}\sum_{n=0}^{\infty}(-1)^n n^k L_n^0(2\alpha) = \frac{1}{(-2i)^k}\lim_{v\to 0}\frac{\partial^k}{\partial v^k}\left(\frac{e^{iv}}{2\cos v}e^{-i\alpha\tan v}\right), \quad (17a)$$

and

$$e^{-\alpha}\sum_{n=0}^{\infty}(-1)^n (n+1)^k e^{-2i(n+1)v}L_n^1(2\alpha) = \frac{1}{(-2i)^k}\lim_{v\to 0}\frac{\partial^k}{\partial v^k}\left(\frac{1}{4\cos^2 v}e^{-i\alpha\tan v}\right). \quad (17b)$$

Equations (17) provide the expression for the fermion propagator up to any given order in $|q_f B|$. For instance, up to $\mathcal{O}(B^6)$, we obtain

$$\begin{aligned}iS(p) &\simeq i\frac{m_f + \not{p}}{p^2 - m_f^2} - |q_f B|\gamma^1\gamma^2\frac{m_f + \not{p}_\parallel}{(p^2 - m_f^2)^2}\mathrm{sign}(q_f B)\\ &-2i|q_f B|^2\frac{(m_f^2 - p_\parallel^2)\not{p}_\perp + p_\perp^2(m_f + \not{p}_\parallel)}{(p^2 - m_f^2)^4}\\ &+2|q_f B|^3\gamma^1\gamma^2\frac{(m_f + \not{p}_\parallel)(p_\parallel^2 + 3p_\perp^2 - m_f^2)}{(p^2 - m_f^2)^5}\mathrm{sign}(q_f B)\\ &-8i|q_f B|^4\frac{(2p_\parallel^2 + 3p_\perp^2 - 2m_f^2)}{(p^2 - m_f^2)^7}\\ &\times\left[(m_f^2 - p_\parallel^2)\not{p}_\perp + p_\perp^2(m_f + \not{p}_\parallel)\right]\\ &-8|q_f B|^5\gamma^1\gamma^2\frac{(m_f + \not{p}_\parallel)}{(p^2 - m_f^2)^8}\mathrm{sign}(q_f B)\\ &\times\left[18p_\perp^2(p_\parallel^2 - m_f^2) + 2(m_f^2 - p_\parallel^2)^2 + 15p_\perp^4\right]\\ &+16i|q_f B|^6\frac{\left[(m_f^2 - p_\parallel^2)\not{p}_\perp + p_\perp^2(m_f + \not{p}_\parallel)\right]}{(p^2 - m_f^2)^{10}}\\ &\times\left[78p_\perp^2(p_\parallel^2 - m_f^2) + 17(m_f^2 - p_\parallel^2)^2 + 45p_\perp^4\right]. \end{aligned} \quad (18)$$

For details see Appendix A. Equation (18) coincides with Eq. (10). The computation also reveals that the approximation is valid provided $|q_f B| < m_f^2$, as can be seen from Eq. (14) where, in order for the geometric series to converge, independent of the value of $p_\parallel^2$, this condition needs to be satisfied. The above is clearer when $p_\parallel^2$ is continued to Eucledian values $p_\parallel^2 \to -p_\parallel^E$, which is a step previous to using this propagator in the context of finite temperature calculations using the Matsubara formalism.

3 Gluon polarization tensor

With the expression for a charged fermion propagator in powers of the field strength at hand, we are in the position to compute the gluon polarization tensor. At one-loop order, this tensor is obtained from the Feynman diagram depicted

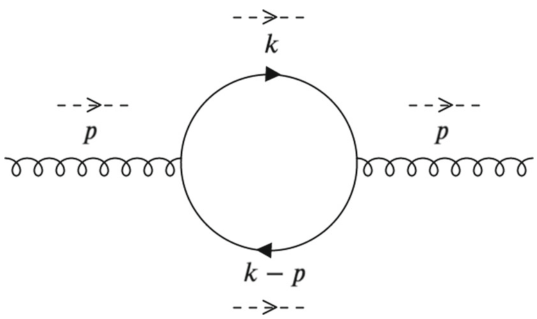

Fig. 1 One-loop diagram representing the gluon polarization tensor

in Fig. 1, where the loop is made out of a quark-antiquark pair. Its explicit expression is

$$i\Pi^{\mu\nu}_{ab}(p) = -\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\{igt_b\gamma^\nu iS(k)igt_a\gamma^\mu iS(k-p)\} + \mathrm{C.C.}, \tag{19}$$

where $igt_a\gamma^\mu$ is the QCD quark-gluon vertex. The global negative sign comes from the fermion loop and C.C. refers to the charge conjugate contribution. Here, we restrict ourselves to using the expansion for iS only up to $\mathcal{O}(B^2)$, using either Eq. (18) or Eq. (10), namely,

$$\begin{aligned} iS(k) &= i\frac{\not k + m_f}{k^2-m^2} - |q_fB|\frac{\gamma^1\gamma^2(\not k_\parallel + m_f)}{(k^2-m_f^2)^2}\mathrm{sign}(q_fB) \\ &\quad - |q_fB|^2\frac{2ik_\perp^2}{(k^2-m_f^2)^4}\left[m_f + \not k_\parallel + \not k_\perp\left(\frac{m_f^2-k_\parallel^2}{k_\perp^2}\right)\right] \\ &\equiv iS^{(0)}(k) + iS^{(1)}(k) + iS^{(2)}(k). \end{aligned} \tag{20}$$

Gauge invariance requires that the gluon polarization tensor be transverse. However, the breaking of Lorentz symmetry makes this tensor to split into three transverse structures [42,43]. The most general symmetric tensor to express the polarization tensor in the presence of a magnetic field can be constructed out of combinations of the four possible independent tensors

$$p^\mu p^\nu,\ b^\mu b^\nu,\ p^\mu b^\nu + p^\nu b^\mu,\ g^{\mu\nu} \tag{21}$$

where b^μ represents the direction of the magnetic field. However, notice that in QCD, $\Pi^{\mu\nu}$ must satisfy the generalized Ward-Takahashi identity namely, the transversality condition

$$p_\mu p_\nu \Pi^{\mu\nu} = 0. \tag{22}$$

Therefore, since the above equation implies a relation between the coefficients of the tensors to express $\Pi^{\mu\nu}$, only three transverse tensors turn out to be independent. Indeed, let us assume that $\Pi^{\mu\nu}$ can be written as

$$\Pi^{\mu\nu} = a\ A^{\mu\nu} + b\ B^{\mu\nu} + c\ C^{\mu\nu} + d\ D^{\mu\nu}. \tag{23}$$

Gauge invariance implies that

$$\begin{aligned} p_\mu p_\nu \Pi^{\mu\nu} &= a\left(p_\mu p_\nu A^{\mu\nu}\right) + b\left(p_\mu p_\nu B^{\mu\nu}\right) \\ &\quad + c\left(p_\mu p_\nu C^{\mu\nu}\right) + d\left(p_\mu p_\nu D^{\mu\nu}\right) = 0. \end{aligned} \tag{24}$$

This means that only three out of the four coefficients (a, b, c, d) are independent. Therefore, the tensor structure that multiplies the coefficient chosen as not independent can be distributed among the rest of the structures to result in only three of them being needed to span the whole tensor $\Pi^{\mu\nu}$. A convenient basis to express the polarization tensor is such that each element is chosen as transverse, in such a way that gauge invariance be satisfied already as

$$p_\mu \Pi^{\mu\nu} = 0. \tag{25}$$

This choice has the advantage that the basis can be used to express the polarization tensor either in QCD or in QED. We chose the orthonormal basis

$$\begin{aligned} \mathcal{P}^{\mu\nu}_\parallel &= g^{\mu\nu}_\parallel - \frac{p^\mu_\parallel p^\nu_\parallel}{p^2_\parallel}, \\ \mathcal{P}^{\mu\nu}_\perp &= g^{\mu\nu}_\perp + \frac{p^\mu_\perp p^\nu_\perp}{p^2_\perp}, \\ \mathcal{P}^{\mu\nu}_0 &= g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \mathcal{P}^{\mu\nu}_\parallel - \mathcal{P}^{\mu\nu}_\perp. \end{aligned} \tag{26}$$

As explicitly discussed in Ref. [39], this choice of basis is appropriate to satisfy the gauge invariance condition.

Nevertheless, the explicit computation reveals that the polarization tensor exhibits in addition a dependence proportional to the non-transverse tensor structures $g^{\mu\nu}_\parallel$ and $g^{\mu\nu}_\perp$, whose coefficients do not obviously vanish. Thus, the tensor needs to be written in general as

$$\Pi^{\mu\nu} = P_\parallel\mathcal{P}^{\mu\nu}_\parallel + P_\perp\mathcal{P}^{\mu\nu}_\perp + P_0\mathcal{P}^{\mu\nu}_0 + A_1 g^{\mu\nu}_\parallel + A_2 g^{\mu\nu}_\perp. \tag{27}$$

In this work we show that the coefficients A_1 and A_2 do vanish, and thus that the gluon polarization tensor obtained using the fermion propagator expanded in powers of $|q_fB|$ up to second order is indeed a transverse tensor. This result was also obtained in Ref. [39] from an expansion to the same order in $|q_fB|$ of the general expression for the polarization tensor in the presence of a magnetic field of arbitrary intensity.

When computing the gluon polarization tensor at one-loop order, the internal quark/antiquark lines are described by a fermion propagator given by Eq. (20). Since a diagram with a single external field insertion vanishes, as required by Furry's theorem, the first nontrivial magnetic contribution to the gluon polarization tensor turns out to be of order $\mathcal{O}(B^2)$. The relevant Feynman diagrams to be computed are depicted in Fig. 2. The photon lines represent the coupling of the external magnetic field to the quark-antiquark pair in the loop. There are two kinds of contributions at order $\mathcal{O}(B^2)$:

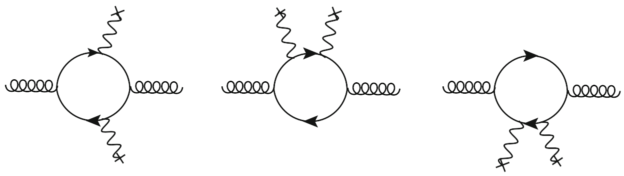

Fig. 2 Feynman diagrams contributing to the one-loop gluon polarization tensor in the weak field limit to second order in B

The first one comes from the product of the linear terms in Eq. (20), and the second one results from the product of the vacuum and the quadratic terms. We call the first term $\Pi^{\mu\nu}_{(1,1)}$ and the second term $\Pi^{\mu\nu}_{(2,0)} + \Pi^{\mu\nu}_{(0,2)}$. $\Pi^{\mu\nu}_{(1,1)}$ is its own C.C. and thus there is no need that its C.C. is added up. On the other hand, $\Pi^{\mu\nu}_{(2,0)}$ and $\Pi^{\mu\nu}_{(0,2)}$ are the C.C. of each other and thus both contributions need to be taken into account. Given the subtleties involved in the calculation, we now proceed to compute in detail each one of these terms.

3.1 $\Pi^{\mu\nu}_{(1,1)}$ contribution

The contribution from $\Pi^{\mu\nu}_{(1,1)}$ can written as

$$
\begin{aligned}
i\Pi^{\mu\nu}_{(1,1)} &= -\int \frac{d^4k}{(2\pi)^4} \frac{\text{Tr}\left\{igt_b\gamma^\nu iS^{(1)}(k-p)igt_a\gamma^\mu iS^{(1)}(k)\right\}}{(k^2-m_f^2)^2\left[(k-p)^2-m_f^2\right]^2} \\
&= 2\left|q_fB\right|^2 g^2 \\
&\int \frac{d^4k}{(2\pi)^4} \frac{k_\parallel^\nu(k_\parallel^\mu - p_\parallel^\mu) + k_\parallel^\mu(k_\parallel^\nu - p_\parallel^\nu) - (g_\parallel^{\mu\nu} - g_\perp^{\mu\nu})\left(k_\parallel\cdot(k_\parallel - p_\parallel) - m_f^2\right)}{(k^2-m_f^2)^2\left[(k-p)^2-m_f^2\right]^2}.
\end{aligned}
\tag{28}
$$

To carry out the calculation, we introduce two Schwinger parameters x_1 and x_2, using the general expression

$$
\frac{1}{k^2-m^2} = -\int_0^\infty dx\, e^{x(k^2-m^2)}. \tag{29}
$$

Thus, using Eq. (29), we rewrite the denominators in Eq. (28) as

$$
\begin{aligned}
&\frac{1}{(k^2-m_f^2)^2\left[(k-p)^2-m_f^2\right]^2} \\
&= \frac{1}{(k^2-m_1^2)^2\left[(k-p)^2-m_2^2\right]^2} \\
&= \frac{\partial}{\partial m_1^2}\frac{\partial}{\partial m_2^2}\int d^2x\, e^{x_2(k^2-m_1^2)}e^{x_1((k-p)^2-m_2^2)},
\end{aligned}
\tag{30}
$$

where in order to exponentiate the denominators in a symmetric manner, we have distinguished the mass in each of the original factors to take the derivatives. We later bring the masses back to be the same. Equation (28) can therefore be written as

$$
\begin{aligned}
i\Pi^{\mu\nu}_{(1,1)} &= 2\left|q_fB\right|^2 g^2 \frac{\partial}{\partial m_1^2}\frac{\partial}{\partial m_2^2}\int\frac{d^4k}{(2\pi)^4} \\
&\times\int d^2x\, \exp\Big[(x_1+x_2)\left(k-\frac{x_1}{x_1+x_2}p\right)^2 \\
&+\frac{x_1x_2}{x_1+x_2}p^2 - (m_1^2x_1+m_2^2x_2)\Big] \\
&\times\Big\{k_\parallel^\nu(k_\parallel^\mu - p_\parallel^\mu) + k_\parallel^\mu(k_\parallel^\nu - p_\parallel^\nu) - (g_\parallel^{\mu\nu} - g_\perp^{\mu\nu}) \\
&\times\Big[k_\parallel\cdot(k_\parallel-p_\parallel) - m_f^2\Big]\Big\}.
\end{aligned}
\tag{31}
$$

In order to perform the integral over the internal momenta, we make the change of variable

$$
l = \left(k - \frac{x_1}{x_1+x_2}p\right), \tag{32}
$$

and by replacing $m_1 = m_2 = m_f$ we obtain

$$
\begin{aligned}
i\Pi^{\mu\nu}_{(1,1)} &= 2\left|q_fB\right|^2 g^2 \int\frac{d^4l}{(2\pi)^4}\int d^2x\, x_1x_2 \\
&\times\exp\Big[(x_1+x_2)(l^2-m_f^2) + \frac{x_1x_2}{x_1+x_2}p^2\Big] \\
&\times\Big\{2l_\parallel^\mu l_\parallel^\nu - 2\frac{x_1x_2}{(x_1+x_2)^2}p_\parallel^\mu p_\parallel^\nu - (g_\parallel^{\mu\nu} - g_\perp^{\mu\nu}) \\
&\times\Big[l_\parallel^2 - \frac{x_1x_2}{(x_1+x_2)^2}p_\parallel^2 - m_f^2\Big]\Big\}.
\end{aligned}
\tag{33}
$$

Using the tensor basis in Eq. (26), we can write

$$
p_\parallel^\mu p_\parallel^\nu = -p_\parallel^2\mathcal{P}_\parallel^{\mu\nu} + p_\parallel^2 g_\parallel^{\mu\nu}, \tag{34}
$$

to then obtain

$$
\begin{aligned}
i\Pi^{\mu\nu}_{(1,1)} &= 2\left|q_fB\right|^2 g^2 \int\frac{d^4l}{(2\pi)^4}\int d^2x\, x_1x_2 \\
&\times\exp\Big[(x_1+x_2)(l^2-m_f^2) + \frac{x_1x_2}{x_1+x_2}p^2\Big] \\
&\times\Big[-2l_\parallel^\mu l_\parallel^\nu - 2\frac{x_1x_2}{(x_1+x_2)^2}p_\parallel^2\mathcal{P}_\parallel^{\mu\nu} \\
&+g_\parallel^{\mu\nu}\left(m_f^2 + l_\parallel^2 + \frac{x_1x_2}{(x_1+x_2)^2}p_\parallel^2\right) \\
&-g_\perp^{\mu\nu}\left(m_f^2 + l_\parallel^2 - \frac{x_1x_2}{(x_1+x_2)^2}p_\parallel^2\right)\Big].
\end{aligned}
\tag{35}
$$

Also, to carry out the integration over the parallel components of the internal momentum, we use the substitution

$$
l_\parallel^\mu l_\parallel^\nu \to \frac{1}{2}g_\parallel^{\mu\nu}l_\parallel^2, \tag{36}
$$

after which the integration yields

$$
i\Pi^{\mu\nu}_{(1,1)} = -\frac{2i\left|q_fB\right|^2 g^2}{16\pi^2}\int d^2x\,\frac{x_1x_2}{(x_1+x_2)^4}
$$

$$\times \exp\Big[-(x_1+x_2)m_f^2 + \frac{x_1 x_2}{x_1+x_2}p^2\Big]$$
$$\times\Big\{+g_\parallel^{\mu\nu}\Big[m_f^2(x_1+x_2)^2 - x_1x_2p_\parallel^2\Big]$$
$$+g_\perp^{\mu\nu}\Big[-m_f^2(x_1+x_2)^2-(x_1+x_2)-x_1x_2p_\parallel^2\Big]$$
$$+2x_1x_2p_\parallel^2\mathcal{P}_\parallel^{\mu\nu}\Big\}. \tag{37}$$

Finally, we perform the change of variables $x_1 = s(1-y)$ and $x_2 = sy$ to get

$$i\Pi^{\mu\nu}_{(1,1)} = -\frac{2i\,|q_fB|^2\,g^2}{16\pi^2}\int_0^1 dy\int_0^\infty ds$$
$$\times sy(1-y)\exp\Big[-s(m_f^2+y(y-1)p^2)\Big]$$
$$\times\Big\{+2y(1-y)p_\parallel^2\mathcal{P}_\parallel^{\mu\nu}$$
$$+g_\parallel^{\mu\nu}\Big[m_f^2-y(1-y)p_\parallel^2\Big]$$
$$+g_\perp^{\mu\nu}\Big[-m_f^2-\frac{1}{s}-y(1-y)p_\parallel^2\Big]\Big\}. \tag{38}$$

Equation (38) represents the first contribution to the gluon polarization tensor. We now proceed to compute the remaining pieces.

3.2 $\Pi^{\mu\nu}_{(2,0)}$ and $\Pi^{\mu\nu}_{(0,2)}$ contributions

The remaining contribution to the polarization tensor of order $\mathcal{O}(B^2)$ can be split in two terms, $\Pi^{\mu\nu}_{(2,0)}$ and $\Pi^{\mu\nu}_{(0,2)}$. The first term corresponds to the second Feynman diagram depicted in Fig. 2, and it has the following expression

$$i\Pi^{\mu\nu}_{(2,0)} = -\int\frac{d^4k}{(2\pi)^4}$$
$$\times\mathrm{Tr}\Big\{igt_b\gamma^\nu iS^{(0)}(k-p)igt_a\gamma^\mu iS^{(2)}(k)\Big\}$$
$$= -g^2\,|q_fB|^2\int\frac{d^4k}{(2\pi)^4}$$
$$\times\frac{1}{((k-p)^2-m_f^2)(k^2-m_f^2)^4}$$
$$\times\mathrm{Tr}\Big\{\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta(k-p)_\alpha\big[k_\beta^\perp(k^2-m_f^2)$$
$$-(k_\beta^\parallel-k_\beta^\perp)k_\perp^2\big]-\gamma^\nu\gamma^\mu k_\perp^2 m_f^2\Big\}. \tag{39}$$

We proceed to compute this contribution in the same manner as for $\Pi^{\mu\nu}_{(1,1)}$. We use the relations in Eqs. (29) and (30) to introduce Schwinger parameters and rewrite the denominators. After using Eq. (32) we obtain

$$i\Pi^{\mu\nu}_{(2,0)} = |q_fB|^2\,g^2\int\frac{d^4l}{(2\pi)^4}\int d^2x\exp\Big[(x_1+x_2)(l^2-m_f^2)$$
$$+\frac{x_1x_2}{x_1+x_2}p^2\Big]$$
$$\times\Big[\frac{x_2^2}{2!}\mathrm{Tr}\Big\{\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\Big(l_\perp-\frac{x_2p_\perp}{x_1+x_2}\Big)_\alpha\Big(l_\perp+\frac{x_1p_\perp}{x_1+x_2}\Big)_\beta\Big\}$$
$$-\frac{x_2^3}{3!}\mathrm{Tr}\Big\{\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\Big(l-\frac{x_2p}{x_1+x_2}\Big)_\alpha\Big(l+\frac{x_1p}{x_1+x_2}\Big)_\beta$$
$$\Big(l_\perp+\frac{x_1p_\perp}{x_1+x_2}\Big)^2+\gamma^\nu\gamma^\mu\Big(l_\perp+\frac{x_1p_\perp}{x_1+x_2}\Big)^2m_f^2\Big\}\Big]$$
$$\equiv|q_fB|^2\,g^2\,(I_1^{\mu\nu}+I_2^{\mu\nu}+I_3^{\mu\nu}), \tag{40}$$

where

$$I_1^{\mu\nu} = \frac{1}{2!}\int\frac{d^4l}{(2\pi)^4}\int d^2x\,x_2^2\exp\Big[(x_1+x_2)(l^2-m_f^2)$$
$$+\frac{x_1x_2}{x_1+x_2}p^2\Big]\mathrm{Tr}\Big\{\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\Big(l_\perp-\frac{x_2p_\perp}{x_1+x_2}\Big)_\alpha$$
$$\times\Big(l_\perp+\frac{x_1p_\perp}{x_1+x_2}\Big)_\beta\Big\}, \tag{41a}$$

$$I_2^{\mu\nu} = -\frac{1}{3!}\int\frac{d^4l}{(2\pi)^4}\int d^2x\,x_2^3\exp\Big[(x_1+x_2)(l^2-m_f^2)$$
$$+\frac{x_1x_2}{x_1+x_2}p^2\Big]\mathrm{Tr}\Big\{\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\Big(l-\frac{x_2p}{x_1+x_2}\Big)_\alpha$$
$$\times\Big(l+\frac{x_1p}{x_1+x_2}\Big)_\beta\Big(l_\perp+\frac{x_1p_\perp}{x_1+x_2}\Big)^2\Big\}, \tag{41b}$$

and

$$I_3^{\mu\nu} = -\frac{m_f^2}{3!}\int\frac{d^4l}{(2\pi)^4}\int d^2x\,x_2^3\exp\Big[(x_1+x_2)(l^2-m_f^2)$$
$$+\frac{x_1x_2}{x_1+x_2}p^2\Big]\mathrm{Tr}\Big\{\gamma^\nu\gamma^\mu\Big(l_\perp+\frac{x_1p_\perp}{x_1+x_2}\Big)^2\Big\}. \tag{41c}$$

Taking the trace in Eqs. (41) and integrating over the internal momenta, discarding odd powers of l, and using the tensor basis in Eq. (26) we can write

$$I_1^{\mu\nu} = \Big(\frac{i}{8\pi^2}\Big)\int d^2x\,x_2^2\exp\Big[-(x_1+x_2)m_f^2+\frac{x_1x_2}{x_1+x_2}p^2\Big]$$
$$\times\Big[\frac{g_\parallel^{\mu\nu}}{(x_1+x_2)^3}+\frac{x_1x_2}{(x_1+x_2)^4}\Big(p^2\mathcal{P}_0^{\mu\nu}-p_\perp\mathcal{P}_\parallel^{\mu\nu}$$
$$+(p_\parallel^2-2p_\perp^2)\mathcal{P}_\perp^{\mu\nu}-g_\perp^{\mu\nu}p^2\Big)\Big], \tag{42a}$$

$$I_2^{\mu\nu} = -\Big(\frac{i}{8\pi^2}\Big)\frac{1}{3}\int d^2x\,x_2^3\exp\Big[-(x_1+x_2)m_f^2+\frac{x_1x_2}{x_1+x_2}p^2\Big]$$
$$\times\Big\{g_\parallel^{\mu\nu}\Big[\frac{2}{(x_1+x_2)^4}+\frac{x_1^2}{(x_1+x_2)^5}p_\perp^2\Big]$$
$$-g_\perp^{\mu\nu}\Big[-\frac{1}{(x_1+x_2)^4}-\frac{x_1^2}{(x_1+x_2)^5}p_\perp^2\Big]$$
$$-\frac{x_1x_2}{(x_1+x_2)^2}\Big[\frac{1}{(x_1+x_2)^3}+\frac{x_1^2}{(x_1+x_2)^4}p_\perp^2\Big]$$
$$\Big[g^{\mu\nu}p^2-2p^2\Big(\mathcal{P}_0^{\mu\nu}+\mathcal{P}_\parallel^{\mu\nu}+\mathcal{P}_\perp^{\mu\nu}\Big)\Big]$$

$$+\frac{x_1(x_1-x_2)}{(x_1+x_2)^5}\left[g_\perp^{\mu\nu}p^2 -p^2\mathcal{P}_0^{\mu\nu}+p_\perp^2\mathcal{P}_\parallel^{\mu\nu}-(p_\parallel^2-2p_\perp^2)\mathcal{P}_\perp^{\mu\nu}\right]\Bigg\}, \quad (42b)$$

and

$$I_3^{\mu\nu}=-\left(\frac{i}{8\pi^2}\right)\frac{1}{3}g^{\mu\nu}m_f^2\int d^2x\, x_2^3\exp\left[-(x_1+x_2)m_f^2+\frac{x_1x_2}{x_1+x_2}p^2\right]\left[\frac{1}{(x_1+x_2)^3}+\frac{x_1^2}{(x_1+x_2)^4}p_\perp^2\right]. \quad (42c)$$

Adding up the results for $I_1^{\mu\nu}$, $I_2^{\mu\nu}$, and $I_3^{\mu\nu}$, we can write $\Pi_{(2,0)}^{\mu\nu}$ as

$$\begin{aligned}
i\Pi_{(2,0)}^{\mu\nu}&=-\left(\frac{i}{8\pi^2}\right)|q_fB|^2g^2\int d^2x\exp\left[-(x_1+x_2)m_f^2+\frac{x_1x_2}{x_1+x_2}p^2\right]\\
&\times\Bigg\{-x_2^2\Bigg[\frac{g_\parallel^{\mu\nu}}{(x_1+x_2)^3}+\frac{x_1x_2}{(x_1+x_2)^4}\left(p^2\mathcal{P}_0^{\mu\nu}-p_\perp\mathcal{P}_\parallel^{\mu\nu}+(p_\parallel^2-2p_\perp^2)\mathcal{P}_\perp^{\mu\nu}-g_\perp^{\mu\nu}p^2\right)\Bigg]\\
&+\frac{x_2^3}{3}\Bigg[g_\parallel^{\mu\nu}\left(\frac{2}{(x_1+x_2)^4}+\frac{x_1^2}{(x_1+x_2)^5}p_\perp^2\right)-g_\perp^{\mu\nu}\left(-\frac{1}{(x_1+x_2)^4}-\frac{x_1^2}{(x_1+x_2)^5}p_\perp^2\right)\\
&-\frac{x_1x_2}{(x_1+x_2)^2}g^{\mu\nu}p^2\left(\frac{1}{(x_1+x_2)^3}+\frac{x_1^2}{(x_1+x_2)^4}p_\perp^2\right)\\
&+\frac{2x_1x_2}{(x_1+x_2)^2}p^2\left(\mathcal{P}_0^{\mu\nu}+\mathcal{P}_\parallel^{\mu\nu}+\mathcal{P}_\perp^{\mu\nu}\right)\left(\frac{1}{(x_1+x_2)^3}+\frac{x_1^2}{(x_1+x_2)^4}p_\perp^2\right)\\
&+\frac{x_1(x_1-x_2)}{(x_1+x_2)^5}\left(g_\perp^{\mu\nu}p^2-p^2\mathcal{P}_0^{\mu\nu}+p_\perp^2\mathcal{P}_\parallel^{\mu\nu}-(p_\parallel^2-2p_\perp^2)\mathcal{P}_\perp^{\mu\nu}\right)\\
&+g^{\mu\nu}m_f^2\left(\frac{1}{(x_1+x_2)^3}+\frac{x_1^2}{(x_1+x_2)^4}p_\perp^2\right)\Bigg]\Bigg\}, \quad (43)
\end{aligned}$$

The full expression for the gluon polarization tensor up to $\mathcal{O}(B^2)$ is obtained after including the contribution from the third Feynman diagram depicted in Fig. 2 corresponding to the term $\Pi_{(0,2)}^{\mu\nu}$. Notice, however, that this last term can be obtained from Eq. (43) after the exchange $x_1\leftrightarrow x_2$. Hence, $\Pi_{(2,0)}^{\mu\nu}$ and $\Pi_{(0,2)}^{\mu\nu}$ contributions can be written together, so that, after introducing the change of variables $x_1=s(1-y)$ and $x_2=sy$, we obtain:

$$\begin{aligned}
i\Pi_{(2,0)}^{\mu\nu}+i\Pi_{(0,2)}^{\mu\nu}&=-\frac{2}{3}\left(\frac{i}{16\pi^2}\right)g^2|q_fB|^2\int_0^1dy\int_0^\infty dss\exp\left[-s(m_f^2+y(y-1)p^2)\right]\\
&\times\Bigg\{g_\parallel^{\mu\nu}\Bigg[-\frac{1}{s}+sy^3(y-1)^3p_\parallel^2p_\perp^2-sy^3(y-1)^3p_\perp^4\\
&-y(y-1)(1+2y(y-1))p_\perp^2\\
&+y(y-1)(1+2y(y-1))p_\parallel^2\\
&+m_f^2\left(1-3y(y-1)+sy^2(y-1)^2p_\perp^2\right)\Bigg]\\
&+g_\perp^{\mu\nu}\Bigg[\frac{1+3y(y-1)}{s}\\
&+sy^3(y-1)^3p_\parallel^2p_\perp^2-sy^3(y-1)^3p_\perp^4\\
&+y(y-1)p_\perp^2+y(1+y^2(y-2))p_\parallel^2\\
&+m_f^2\left(1-3y(y-1)+sy^2(y-1)^2p_\perp^2\right)\Bigg]\\
&+\mathcal{P}_0^{\mu\nu}\left[-4y^2(y-1)^2p^2-2sy^3(y-1)^3p^2p_\perp^2\right]\\
&+\mathcal{P}_\parallel^{\mu\nu}\Big[-2y(y-1)(1+3y(y-1))p_\parallel^2\\
&+4y^2(y-1)^2p_\perp^2-2sy^3(y-1)^3p^2p_\perp^2\Big]\\
&+\mathcal{P}_\perp^{\mu\nu}\Big[-4y^2(y-1)^2p_\parallel^2+2y(1+y^2(y-2))p_\perp^2\\
&-2sy^3(y-1)^3p^2p_\perp^2\Big]\Bigg\}. \quad (44)
\end{aligned}$$

Adding the contributions from $\Pi_{(1,1)}^{\mu\nu}$, $\Pi_{(2,0)}^{\mu\nu}$, and $\Pi_{(0,2)}^{\mu\nu}$, we finally obtain

$$\begin{aligned}
i\Pi^{\mu\nu}&=-\frac{2}{3}\left(\frac{i}{16\pi^2}\right)g^2|q_fB|^2\int_0^1dy\int_0^\infty dss\\
&\exp\left[-s(m_f^2+y(y-1)p^2)\right]\\
&\times\Bigg\{g_\parallel^{\mu\nu}\Bigg[-\frac{1}{s}+sy^3(y-1)^3p_\parallel^2p_\perp^2\\
&-sy^3(y-1)^3p_\perp^4-y(y-1)(1+2y(y-1))p_\perp^2\\
&+y(y-1)p_\parallel^2+m_f^2\left(1+sy^2(y-1)^2p_\perp^2\right)\Bigg]\\
&+g_\perp^{\mu\nu}\Bigg[\frac{1+6y(y-1)}{s}+sy^3(y-1)^3p_\parallel^2p_\perp^2\\
&-sy^3(y-1)^3p_\perp^4+y(y-1)p_\perp^2\\
&-y(y-1)(1+2y(y-1))p_\parallel^2+m_f^2\left(1+6y(y-1)+sy^2(y-1)^2p_\perp^2\right)\Bigg]
\end{aligned}$$

$$+\mathcal{P}_0^{\mu\nu}\Big[-4y^2(y-1)^2p^2-2sy^3(y-1)^3p^2p_\perp^2\Big]$$
$$+\mathcal{P}_\parallel^{\mu\nu}\Big[-2y(y-1)p_\parallel^2+4y^2(y-1)^2p_\perp^2$$
$$-2sy^3(y-1)^3p^2p_\perp^2\Big]$$
$$+\mathcal{P}_\perp^{\mu\nu}\Big[-4y^2(y-1)^2p_\parallel^2+2y(1+y^2(y-2))p_\perp^2$$
$$-2sy^3(y-1)^3p^2p_\perp^2\Big]\Big\}. \quad (45)$$

Notice that Eq. (45) contains terms proportional to the tensors $g_\parallel^{\mu\nu}$ and $g_\perp^{\mu\nu}$. Their coefficients are the A_1 and A_2 factors in Eq. (27), respectively. Nevertheless, it is easy to show that the coefficient A_1 vanishes when carrying out the integration over s. In a similar fashion, A_2 also vanishes upon integration over both s and y. Therefore we can finally write

$$i\Pi^{\mu\nu}=-\frac{2}{3}\left(\frac{i}{16\pi^2}\right)g^2\left|q_fB\right|^2\int_0^1dy\int_0^\infty ds\,s$$
$$\exp\Big[-s(m_f^2+y(y-1)p^2)\Big]$$
$$\times\Big\{\mathcal{P}_0^{\mu\nu}\Big[-4y^2(y-1)^2p^2-2sy^3(y-1)^3p^2p_\perp^2\Big]$$
$$+\mathcal{P}_\parallel^{\mu\nu}\Big[-2y(y-1)p_\parallel^2+4y^2(y-1)^2p_\perp^2$$
$$-2sy^3(y-1)^3p^2p_\perp^2\Big]$$
$$+\mathcal{P}_\perp^{\mu\nu}\Big[-4y^2(y-1)^2p_\parallel^2+2y(1+y^2(y-2))p_\perp^2$$
$$-2sy^3(y-1)^3p^2p_\perp^2\Big]\Big\}. \quad (46)$$

For the momentum range $0\le p^2\le 4m_f^2$, Eq. (46) lends itself to perform the integrations analytically. Thus, we finally obtain the explicit coefficients of the three basis tensor structures shown in Eq. (26). These are given by

$$P_0=\frac{ig^2\left|q_fB\right|^2}{6\pi^2}\Bigg\{\frac{p^2-6m^2}{p^2\left(p^2-4m^2\right)}$$
$$+\frac{\left(p^2-10m^2\right)\left(p^2-3m^2\right)}{p^4\left(p^2-4m^2\right)^2}p_\perp^2$$
$$+\Bigg[\frac{8m^2\left(p^2-3m^2\right)}{(p^2)^{3/2}\left(4m^2-p^2\right)^{3/2}}$$
$$-\frac{12m^2\left(10m^4+(p^2-6m^2)p^2\right)}{\left(4m^2-p^2\right)^{5/2}(p^2)^{5/2}}p_\perp^2\Bigg]$$
$$\arctan\left(\sqrt{\frac{p^2}{4m_f^2-p^2}}\right)\Bigg\}, \quad (47a)$$

$$P_\parallel=\frac{ig^2\left|q_fB\right|^2}{6\pi^2}\Bigg\{-\frac{p_\parallel^2}{\left(4m^2-p^2\right)p^2}-\frac{(p^2-6m^2)p_\perp^2}{p^4\left(p^2-4m^2\right)}$$
$$+\frac{\left(p^2-10m^2\right)\left(p^2-3m^2\right)p_\perp^2}{p^4\left(p^2-4m^2\right)^2}$$
$$+\Bigg[\frac{-2\left(p^2-2m^2\right)p_\parallel^2}{\left(4m^2-p^2\right)^{3/2}(p^2)^{3/2}}$$
$$-\frac{8m^2\left(p^2-3m^2\right)p_\perp^2}{\left(4m^2-p^2\right)^{3/2}(p^2)^{5/2}}$$
$$-\frac{12m^2\left(10m^4+(p^2-6m^2)p^2\right)}{\left(4m^2-p^2\right)^{5/2}(p^2)^{5/2}}p_\perp^2\Bigg]$$
$$\arctan\left(\sqrt{\frac{p^2}{4m_f^2-p^2}}\right)\Bigg\}, \quad (47b)$$

$$P_\perp=\frac{ig^2\left|q_fB\right|^2}{6\pi^2}\Bigg\{\frac{\left(p^2-6m^2\right)p_\parallel^2}{p^4\left(p^2-4m^2\right)}-\frac{\left(6m^2+p^2\right)p_\perp^2}{2\left(4m^2-p^2\right)p^4}$$
$$+\frac{\left(p^2-10m^2\right)\left(p^2-3m^2\right)p_\perp^2}{p^4\left(p^2-4m^2\right)^2}$$
$$+\Bigg[\frac{8m^2\left(p^2-3m^2\right)p_\parallel^2}{\left(4m^2-p^2\right)^{3/2}(p^2)^{5/2}}$$
$$+\frac{2\left(6m^4-p^4\right)p_\perp^2}{\left(4m^2-p^2\right)^{3/2}(p^2)^{5/2}}$$
$$-\frac{12m^2\left(10m^4+(p^2-6m^2)p^2\right)}{\left(4m^2-p^2\right)^{5/2}(p^2)^{5/2}}p_\perp^2\Bigg]$$
$$\arctan\left(\sqrt{\frac{p^2}{4m_f^2-p^2}}\right)\Bigg\}. \quad (47c)$$

It is worth noticing that the result in Eqs. (47a), (47b) and (47c) coincide with the $\mathcal{O}(B^2)$ found in Ref. [39].

4 Gluon dispersion relation and polarizations

The dispersion properties and polarization strength of the propagating gluon modes are encoded in the coefficients of the tensor structures in Eq. (46). In order to extract these properties, we proceed first to find the possible magnetic field-induced modification to the gluon mass and dispersion relations. For this purpose, henceforth, we use $\alpha_s=g^2/4\pi=0.3$ in the analysis. Figure 3 shows plots representing the solutions of the equation

$$p_0^2=P_i(p_0)\Big|_{p_\perp=0,\,p_3=0} \quad (48)$$

for $i=(0,\parallel,\perp)$ as functions of the ratio p_0/m_f and for a fixed value of the ratio $|q_fB|/m_f^2=0.5$. The intercept of the curves on the left- and right-hand side of Eq. (48) corresponds to the gluon magnetic mass for each mode. The

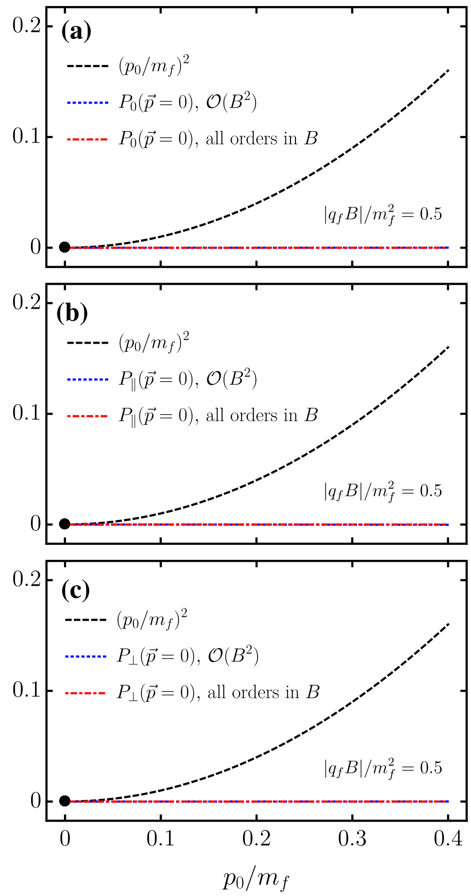

Fig. 3 The gluon magnetic mass obtained from the coefficients (**a**) P_0, (**b**) $P_\parallel$ and (**c**) $P_\perp$ found as the intersection of the curves corresponding to the expressions on the left- and right-hand sides of Eq. (48). Shown are also the numerical calculations to all orders in B which confirm that the only solution, indicated by the full circle, happens for $p_0 = 0$

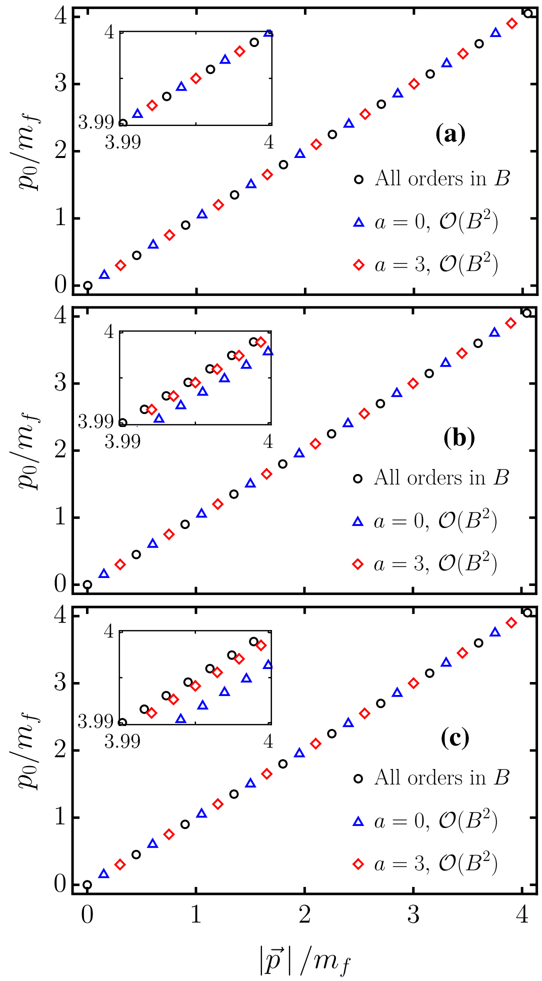

Fig. 4 Dispersion relation obtained for the modes corresponding to (**a**) P_0, (b) $P_\parallel$, and (**c**) $P_\perp$ for two different values of the parameter a computed up to $\mathcal{O}(B^2)$ and to all orders in B for $|q_f B|/m_f^2 = 0.5$. The calculation to all orders corresponds to motion along the light cone for all three modes. In the case of (**a**), the approximation provides the exact solution. For (**b**) and (**c**), slight deviations with respect to the exact result are found and the light cone is approached to from below

figure shows the case where these coefficients are computed up to $\mathcal{O}(B^2)$ as well as to all orders in B, using the findings of Ref. [39]. Notice that in the momentum range considered, the only solution for the gluon mass is $p_0 = 0$.

We now discuss the dispersion relation of each mode. Due to the loss of Lorentz invariance, the coefficients P_i are in principle functions of the two variables $p_\parallel^2$ and $p_\perp^2$ and not of p_0^2 and $|\mathbf{p}|^2 = p_\perp^2 + p_3^2$, which could cast doubts on whether or not the gluons move along the light cone, as should correspond to a massless excitation. In order to check if this is the case, we parametrize the relation between p_3^2 and $p_\perp^2$ as $p_3^2 = a\, p_\perp^2$. Written in this fashion, the case $a = 0$ corresponds to motion in the transverse plane, whereas the case $a \to \infty$ corresponds to motion along the magnetic field direction.

Figure 4 shows the dispersion relation for each mode computed with $|q_f B|/m_f^2 = 0.5$ and two values of a as a function of $|\mathbf{p}| = \sqrt{(1+a)p_\perp^2}$ up to $\mathcal{O}(B^2)$, using the coefficients found in Eqs. (47a), (47b) and (47c). Notice that the calculation to all orders corresponds to motion along the light cone for all three modes. In the case of P_0, the approximation provides the exact solution. For $P_\parallel$ and $P_\perp$, slight deviations with respect to the exact result are found and the light cone is approached to from below. The calculation is also independent of the value of a.

As a further example of the properties of the dispersion relations, Fig. 5 shows the solutions for the equation

$$p_0^2 - p^2 = P_i(p_0, p), \tag{49}$$

for the mode $i = 0$, as a function of p_0/m_f for two values of a and two values of $p_\perp$. The plots show the results for the calculation up to $\mathcal{O}(B^2)$ as well as to all orders in B, for $|q_f B|/m_f^2 = 0.5$. In both cases the intercept happens for $p_0/m_f = \sqrt{(1+a)}(p_\perp/m_f)$. The analysis shows that, for

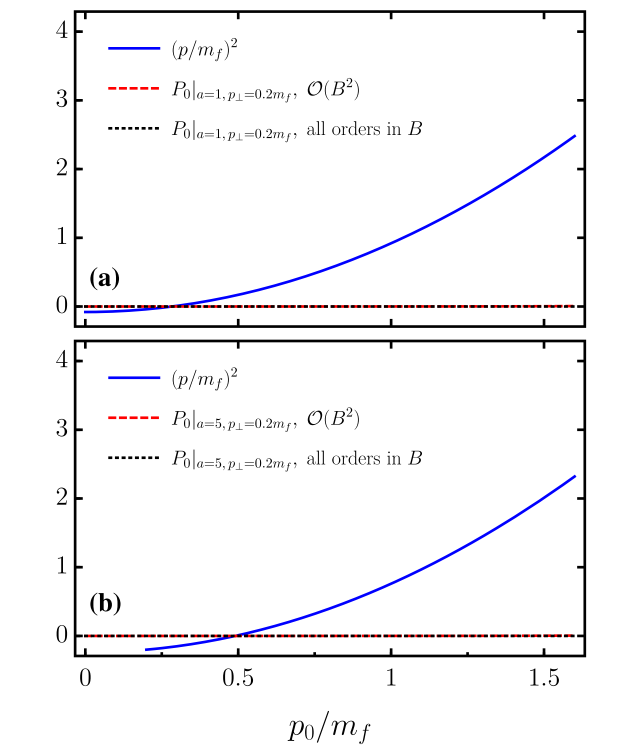

Fig. 5 Solutions for the dispersion relation for the mode $i = 0$ as functions of p_0/m_f for two fixed values of a and two values of $p_\perp$. The plots show the results of the calculation up to $\mathcal{O}(B^2)$ and to all orders in B, for $|q_f B|/m_f^2 = 0.5$, Notice that in both cases the intercept between the solid and dashed or dotted lines happens for $p_0/m_f = \sqrt{(1+a)}(p_\perp/m_f)$, as corresponds to propagation along the light-cone

all practical purposes, one can safely regard gluons as moving along the light-cone even when considering the $\mathcal{O}(B^2)$ calculation.

In order to study the strength of the polarization modes for real gluons, we now proceed to analyse the behavior of Eqs. (47a), (47b) and (47c) when $p_0^2 = |\mathbf{p}|$. Since the particle production threshold condition is also the kinematical limit of applicability of the analytical approximations for the weak field case, we restrict ourselves to showing results only for the real part of the polarization coefficients. Notice that for real gluons $p_\parallel^2 \to p_\perp^2$ and therefore Eqs. (47a), (47b), and (47c) become functions of only $p_\perp^2$ and also that Eq. (47a) vanishes for real gluons with $p^2 = 0$. Figures 6a,b show the normalized coefficients $\hat{P}_\parallel \equiv \left(8\pi^2/g^2m_f^2\right) P_\parallel$ and $\hat{P}_\perp \equiv \left(8\pi^2/g^2m_f^2\right) P_\perp$ as functions of $p_\perp^2/m_f^2$ for three values of the ratio $|q_f B|/m_f^2$ with the on-shell condition. Also, the horizontal axis is restricted to the range $0 \leq p_\perp^2/m_f^2 \leq 4$, where the weak field expressions in Eqs. (47a)–(47c) are valid. The full symbols correspond to the results obtained from Eqs. (47b) and (47c), whereas the open symbols come from the general expressions given in Ref. [39]. Notice that

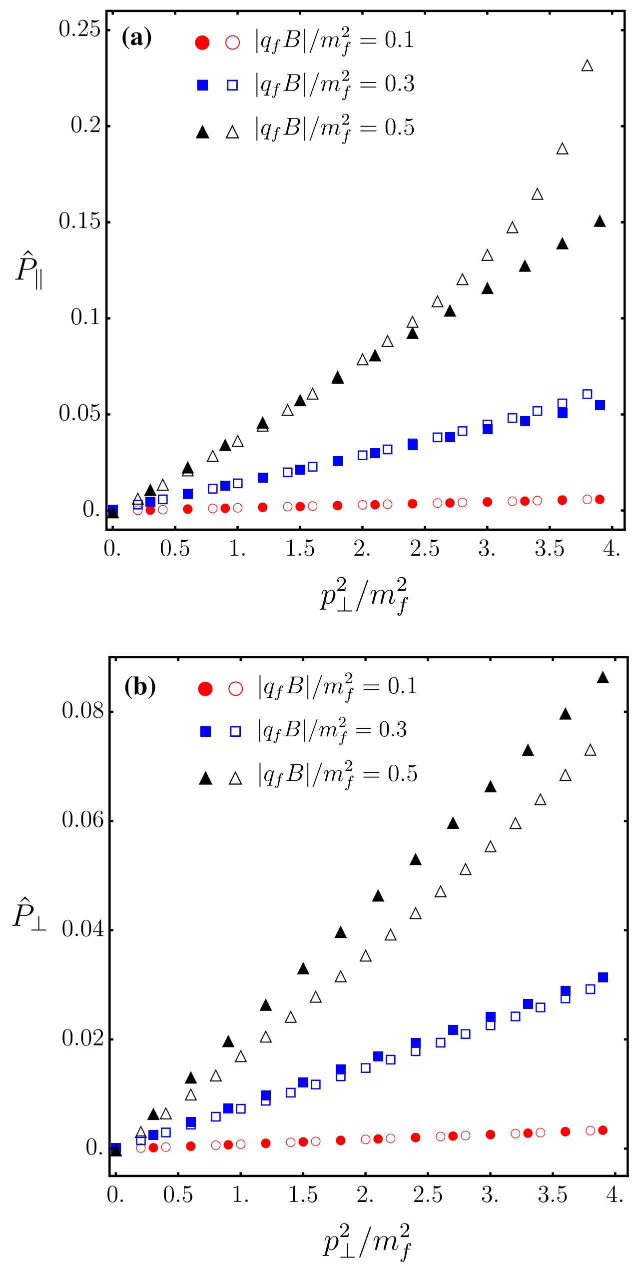

Fig. 6 Normalized tensor coefficients $\hat{P}_{\parallel,\perp}$ as a function of $p_\perp^2/m_f^2$ computed from the weak field expansion given by Eqs. (47b)-(47c) (full symbols) and compared with the general expressions found in Ref. [39] (open symbols). The results are shown for the three values of the ratio $\left|q_f B\right|/m_f^2 = 0.1, 0.3$ and 0.5 and with the on-shell condition $p^2 = 0$

the weak field expansion is accurate for small values of $\left|q_f B\right|/m_f^2$, and also that the deviations are more significant when $p_\perp^2$ increases.

5 Summary and conclusions

In this work we have computed and analysed the properties of the gluon polarization tensor in the presence of a weak, uniform, and constant magnetic field. For this purpose, we derive the fermion propagator for the case when the magnetic field is the smallest energy (squared) scale. We showed two different and alternative methods to derive the propagator. The first one starts from the Schwinger's proper-time representation. In this case we find that it is enough to perform a Taylor expansion to then integrate over the proper time parameter. The second method starts from the propagator written in terms of the Landau Levels representation. This method shows how one can replace the sum over Landau Levels by a series in powers of the magnetic field strength. The condition to be satisfied is that $|q_f B| < m_f^2$. Both methods coincide and the result is illustrated writing the propagator up to $\mathcal{O}(B^6)$.

The expression found in this work for the fermion propagator is in complete agreement with the one reported in Ref. [37]. It is worth mentioning that in a weak field expansion one needs to distinguish the energy scale with respect to which the field is weak. In the present context the energy scales for the comparison are the fermion mass and the gluon momentum components. Therefore, if the field is weak and one wants to explore the properties of the polarization tensor as a function of the gluon momentum, where the latter can also be taken as small, the reference scale is the fermion mass which then cannot be taken as vanishing.

With the expression for the fermion propagator in the weak field approximation at hand, we compute the first non trivial magnetic contribution to the polarization tensor, namely, the $\mathcal{O}(B^2)$ contribution. Since the magnetic field breaks Lorentz symmetry, in order to respect gauge invariance, the polarization tensor is expressed in terms of three tensor structures, $\mathcal{P}_\parallel^{\mu\nu}$, $\mathcal{P}_\perp^{\mu\nu}$, and $\mathcal{P}_0^{\mu\nu}$, which are explicitly transverse. When the computation is explicitly carried out, two other non-physical tensor structures appear, $g_\parallel^{\mu\nu}$ and $g_\perp^{\mu\nu}$, whose coefficients are shown to vanish.

The coefficients of the three physical tensor structures, P_0, $P_\parallel$, and $P_\perp$, are provided explicitly and they coincide with the result at $\mathcal{O}(B^2)$ from the expansion to the same order starting from the expression valid to all orders, reported in Ref. [39]. The agreement between the results in this work and those in Ref. [39] shows that the quantum corrections due to the magnetic field, in the weak field approximation are reliable as the ratio $|q_f B|/m_f^2$ becomes smaller.

We have also studied the dispersion properties and the polarization strength of the propagating gluon modes. Since, the magnetic field breaks Lorentz symmetry, the gluon four-momentum splits into $p_\parallel$ and $p_\perp$ components and thus the polarization tensors become themselves functions of these components. Therefore, it is not obvious that gluons do not develop a magnetic mass and that their dispersion relation is not modified. As we have shown, this is indeed the case and gluons move along the light cone even in the presence of a background magnetic field. This is confirmed by comparing the results at $\mathcal{O}(B^2)$ with the computation performed at all orders using the results from Ref. [39]. We notice that the imaginary parts of the polarization coefficients appear only beyond threshold. Since our analytical solutions to $\mathcal{O}(B^2)$ are valid only up to the threshold value of the gluon's momentum, we cannot study the imaginary parts using this method. We emphasize that any perturbative approach is bound to be valid for a restricted kinematical range. This is the case for the $\mathcal{O}(B^2)$ approximation to the polarization coefficients P_i found in this work. The analysis to find the mass solution shows that, as long as $p_0/m_f \lesssim 1$, the approximate expressions for P_i coincide with the exact expression.

The findings of this work are now suited to be extended to the finite temperature and density case when either one is larger than the field strength. This is work currently being developed and that will be reported elsewhere.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: There is not associated data since all the plot are obtained from the anallytic expression in the manuscript.]

A derivation of Eqs. (18) and (58)

From Eq. (11) three sums can be identified:

$$s_1 = 2ie^{-\alpha}\sum_{n=0}^{\infty}(-1)^n \frac{L_n^0(2\alpha)}{p_\parallel^2 - m_f^2 - 2n\left|q_f B\right|} = \frac{2ie^{-\alpha}}{p_\parallel^2 - m_f^2}\sum_{n=0}^{\infty}(-1)^n \frac{L_n^0(2\alpha)}{1-\frac{2n|q_f B|}{p_\parallel^2 - m_f^2}}, \tag{50a}$$

$$s_2 = -2ie^{-\alpha}\sum_{n=1}^{\infty}(-1)^n \frac{L_{n-1}^0(2\alpha)}{p_\parallel^2 - m_f^2 - 2n\left|q_f B\right|} = -\frac{2ie^{-\alpha}}{p_\parallel^2 - m_f^2}\sum_{n=1}^{\infty}(-1)^n \frac{L_{n-1}^0(2\alpha)}{1-\frac{2n|q_f B|}{p_\parallel^2 - m_f^2}}, \tag{50b}$$

$$s_3 = 4ie^{-\alpha}\sum_{n=1}^{\infty}(-1)^n \frac{L_{n-1}^1(2\alpha)}{p_\parallel^2 - m_f^2 - 2n\left|q_f B\right|} = \frac{4ie^{-\alpha}}{p_\parallel^2 - m_f^2}\sum_{n=1}^{\infty}(-1)^n \frac{L_{n-1}^1(2\alpha)}{1-\frac{2n|q_f B|}{p_\parallel^2 - m_f^2}}, \tag{50c}$$

with $\alpha = p_\perp^2/\left|q_f B\right|$.

The assumption that the magnetic field is the weakest scale allows us to promote the denominator of Eqs. (50) to a geometric series, as it was shown in Eq. (14), which yields two nested sums. The sum in n can be performed using the Eqs. (16) and (17) so that

$$s_1 = \frac{i}{p_\parallel^2 - m_f^2}\sigma_1, \tag{51a}$$

$$s_2 = \frac{i}{p_\parallel^2 - m_f^2}\sigma_2, \tag{51b}$$

and

$$s_3 = -\frac{i}{p_\parallel^2 - m_f^2}\sigma_3, \tag{51c}$$

where

$$\sigma_1 \equiv \lim_{v\to 0}\sum_{k=0}^{\infty}\left(\frac{i\left|q_f B\right|}{p_\parallel^2 - m_f^2}\right)^k \frac{\partial^k}{\partial v^k}\left(\frac{e^{iv}e^{-i\alpha\tan v}}{\cos v}\right), \tag{52a}$$

$$\sigma_2 \equiv \lim_{v\to 0}\sum_{k=0}^{\infty}\left(\frac{i\left|q_f B\right|}{p_\parallel^2 - m_f^2}\right)^k \frac{\partial^k}{\partial v^k}\left(\frac{e^{-iv}e^{-i\alpha\tan v}}{\cos v}\right), \tag{52b}$$

and

$$\sigma_3 \equiv \lim_{v\to 0}\sum_{k=0}^{\infty}\left(\frac{i\left|q_f B\right|}{p_\parallel^2 - m_f^2}\right)^k \frac{\partial^k}{\partial v^k}\left(\frac{e^{-i\alpha\tan v}}{\cos^2 v}\right). \tag{52c}$$

The remaining sums cannot be expressed in closed-form. However, it is possible to find the contribution at the desired order in the parameter space $(p_\parallel^2, p_\perp^2, \left|q_f B\right|, m_f^2)$. By expanding the sum over k and together with the definitions $x = p_\perp^2/(p_\parallel^2 - m_f^2)$ and $\mathfrak{B} = \left|q_f B\right|/(p_\parallel^2 - m_f^2)$, it is straightforward to find the following series:

$$\begin{aligned}
\sigma_1 = &\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\
&+ x^7 + x^8 + x^9 + x^{10} + \cdots\Big) \\
&- \Big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 \\
&+ 8x^7 + 9x^8 + 10x^9 + 11x^{10} + \cdots\Big)\mathfrak{B} \\
&- 2x\Big(1 + 4x + 10x^2 + 20x^3 + 35x^4 \\
&+ 56x^5 + 84x^6 + 120x^7 + 165x^8 \\
&+ 220x^9 + 286x^{10} + \cdots\Big)\mathfrak{B}^2 \\
&+ 2\Big(1 + 8x + 30x^2 + 80x^3 + 175x^4 + 336x^5 \\
&+ 588x^6 + 960x^7 + 1485x^8 + 2200x^9 \\
&+ 3146x^{10} + \cdots\Big)\mathfrak{B}^3 \\
&+ 8x\Big(2 + 17x + 77x^2 + 252x^3 + 672x^4 \\
&+ 1554x^5 + 3234x^6 + 6204x^7 + 11154x^8 + 19019x^9 \\
&+ 31031x^{10} + \cdots\Big)\mathfrak{B}^4 \\
&- 8\Big(2 + 34x + 231x^2 + 1008x^3 \\
&+ 3360x^4 + 9324x^5 + 22638x^6 + 49632x^7 \\
&+ 100386x^8 + 190190x^9 + 341341x^{10} + \cdots\Big)\mathfrak{B}^5 \\
&- 16x\Big(17 + 248x + 1760x^2 + 8480x^3 \\
&+ 31790x^4 + 99704x^5 + 273416x^6 \\
&+ 674960x^7 + 1530815x^8 + 3237520x^9 \\
&+ 6456736x^{10} + \cdots\Big)\mathfrak{B}^6 + \mathcal{O}\left(\mathfrak{B}^7\right),
\end{aligned} \tag{53a}$$

$$\begin{aligned}
\sigma_2 = &\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\
&+ x^7 + x^8 + x^9 + x^{10} + \cdots\Big) \\
&+ \Big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 \\
&+ 8x^7 + 9x^8 + 10x^9 + 11x^{10} + \cdots\Big)\mathfrak{B} \\
&- 2x\Big(1 + 4x + 10x^2 + 20x^3 + 35x^4 \\
&+ 56x^5 + 84x^6 + 120x^7 + 165x^8 \\
&+ 220x^9 + 286x^{10} + \cdots\Big)\mathfrak{B}^2 \\
&- 2\Big(1 + 8x + 30x^2 + 80x^3 + 175x^4 + 336x^5 \\
&+ 588x^6 + 960x^7 + 1485x^8 + 2200x^9 \\
&+ 3146x^{10} + \cdots\Big)\mathfrak{B}^3 \\
&+ 8x\Big(2 + 17x + 77x^2 + 252x^3 + 672x^4 + 1554x^5 \\
&+ 3234x^6 + 6204x^7 + 11154x^8 + 19019x^9 \\
&+ 31031x^{10} + \cdots\Big)\mathfrak{B}^4 \\
&+ 8\Big(2 + 34x + 231x^2 + 1008x^3 + 3360x^4 \\
&+ 9324x^5 + 22638x^6 + 49632x^7 + 100386x^8 \\
&+ 190190x^9 + 341341x^{10} + \cdots\Big)\mathfrak{B}^5 \\
&- 16x\Big(17 + 248x + 1760x^2 + 8480x^3 + 31790x^4 \\
&+ 99704x^5 + 273416x^6 + 674960x^7 + 1530815x^8 \\
&+ 3237520x^9 + 6456736x^{10} + \cdots\Big)\mathfrak{B}^6 + \mathcal{O}\left(\mathfrak{B}^7\right)
\end{aligned} \tag{53b}$$

and

$$\begin{aligned}
\sigma_3 = &\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\
&+ x^7 + x^8 + x^9 + x^{10} + \cdots\Big) \\
&- 2\Big(1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + 84x^6 \\
&+ 120x^7 + 165x^8 + 220x^9 \\
&+ 286x^{10} + \cdots\Big)\mathfrak{B}^2 \\
&+ 8\Big(2 + 17x + 77x^2 + 252x^3 + 672x^4
\end{aligned}$$

$$
\begin{aligned}
&+1554x^5+3234x^6+6204x^7\\
&+11154x^8+19019x^9+31031x^{10}+\cdots\Big)\mathfrak{B}^4\\
&-16\Big(17+248x+1760x^2+8480x^3\\
&+31790x^4+99704x^5+273416x^6\\
&+674960x^7+1530815x^8+3237520x^9\\
&+6456736x^{10}\Big)\mathfrak{B}^6+\mathcal{O}\left(\mathfrak{B}^8\right).
\end{aligned} \tag{53c}
$$

The power series of x in Eqs. (53) can be easily recognized yielding

$$
\begin{aligned}
\sigma_1 = & \frac{1}{1-x}-\frac{\mathfrak{B}}{(1-x)^2}-\frac{2x\mathfrak{B}^2}{(1-x)^4}+\frac{2(3x+1)\mathfrak{B}^3}{(1-x)^5}\\
&+\frac{8x(3x+2)\mathfrak{B}^4}{(1-x)^7}-\frac{8\left(15x^2+18x+2\right)\mathfrak{B}^5}{(1-x)^8}\\
&-\frac{16x\left(45x^2+78x+17\right)\mathfrak{B}^6}{(1-x)^{10}}+\mathcal{O}\left(\mathfrak{B}^7\right),
\end{aligned} \tag{54a}
$$

$$
\begin{aligned}
\sigma_2 = & \frac{1}{1-x}+\frac{\mathfrak{B}}{(1-x)^2}-\frac{2x\mathfrak{B}^2}{(1-x)^4}-\frac{2(3x+1)\mathfrak{B}^3}{(1-x)^5}\\
&+\frac{8x(3x+2)\mathfrak{B}^4}{(1-x)^7}+\frac{8\left(15x^2+18x+2\right)\mathfrak{B}^5}{(1-x)^8}\\
&-\frac{16x\left(45x^2+78x+17\right)\mathfrak{B}^6}{(1-x)^{10}}+\mathcal{O}\left(\mathfrak{B}^7\right),
\end{aligned} \tag{54b}
$$

and

$$
\begin{aligned}
\sigma_3 = & \frac{1}{1-x}-\frac{2\mathfrak{B}^2}{(1-x)^4}+\frac{8(3x+2)\mathfrak{B}^4}{(1-x)^7}\\
&-\frac{16\left(45x^2+78x+17\right)\mathfrak{B}^6}{(1-x)^{10}}+\mathcal{O}\left(\mathfrak{B}^8\right).
\end{aligned} \tag{54c}
$$

The final result is obtained by replacing Eqs. (54c) into Eqs. (51) and Eqs. (50) in such a way that the matrix structures are collected as

$$
\not{p}=\not{p}_\parallel-\not{p}_\perp, \tag{55a}
$$

$$
\mathcal{O}^+ + \mathcal{O}^- = 1, \tag{55b}
$$

and

$$
\mathcal{O}^+ - \mathcal{O}^- = i\,\mathrm{sign}(q_f B)\gamma^1\gamma^2, \tag{55c}
$$

so that at order $\mathfrak{B}^6$ the fermion propagator is

$$
\begin{aligned}
iS(p) =& \frac{i}{p_\parallel^2-m_f^2}\Bigg\{\frac{1}{1-x}\left(\not{p}+m_f\right)\\
&+\Bigg[\frac{\mathfrak{B}}{(1-x)^2}-\frac{2(3x+1)\mathfrak{B}^3}{(1-x)^5}\\
&+\frac{8\left(15x^2+18x+2\right)\mathfrak{B}^5}{(1-x)^8}\Bigg]i\\
&\mathrm{sign}(q_f B)\gamma^1\gamma^2(\not{p}_\parallel+m_f)\\
&-\Bigg[\frac{2\mathfrak{B}^2}{(1-x)^4}-\frac{8(3x+2)\mathfrak{B}^4}{(1-x)^7}\\
&-\frac{16\left(45x^2+78x+17\right)\mathfrak{B}^6}{(1-x)^{10}}\Bigg]\\
&\left(x(\not{p}_\parallel+m_f)-\not{p}_\perp\right)\Bigg\},
\end{aligned} \tag{56}
$$

which after substituting x yields

$$
\begin{aligned}
iS(p) =& \frac{i\left(\not{p}+m_f\right)}{p^2-m_f^2}-\mathcal{G}_1(p,B)\,\mathrm{sign}(q_f B)\gamma^1\gamma^2(\not{p}_\parallel+m_f)\\
&-2i(p_\parallel^2-m_f^2)\,\mathcal{G}_2(p,B)\left[\frac{p_\perp^2}{p_\parallel^2-m_f^2}(\not{p}_\parallel+m_f)-\not{p}_\perp\right],
\end{aligned} \tag{57}
$$

where

$$
\begin{aligned}
\mathcal{G}_1(p,B) \equiv & \frac{|q_f B|}{(p^2-m_f^2)^2}-2\frac{3p_\perp^2+p_\parallel^2-m_f^2}{(p^2-m_f^2)^5}|q_f B|^3\\
&+8\frac{15p_\perp^4+18p_\perp^2(p_\parallel^2-m_f^2)+2(p_\parallel^2-m_f^2)^2}{(p^2-m_f^2)^8}|q_f B|^5,
\end{aligned} \tag{58a}
$$

and

$$
\begin{aligned}
\mathcal{G}_2(p,B) \equiv & \frac{|q_f B|^2}{(p^2-m_f^2)^4}-4\frac{3p_\perp^2-2(p_\parallel^2-m_f^2)}{(p^2-m_f^2)^7}|q_f B|^4\\
&-8\frac{45p_\perp^4+78(p_\parallel^2-m_f^2)+17(p_\parallel^2-m_f^2)^2}{(p^2-m_f^2)^{10}}|q_f B|^6,
\end{aligned} \tag{58b}
$$

so that a rearrangement of terms gives the result shown in Eq. (18).

References

1. L. Adamczyk et al., (STAR Collaboration). Phys. Rev. Lett. **113**, 052302 (2014)
2. D.E. Kharzeev, L.D. McLerran, H.J. Warringa, Nucl. Phys. A **803**, 227–253 (2008)
3. A. Adare et al., (PHENIX Collaboration). Phys. Rev. C **91**, 064904 (2015)
4. J. Adam et al., (ALICE Collaboration). Phys. Lett. B **754**, 235 (2016)
5. A. Adare et al., (PHENIX Collaboration). Phys. Rev. C **98**, 054902 (2018)
6. S. Acharya et al., (ALICE Collaboration). Phys. Rev. C **99**, 024912 (2019)

7. A. Adare et al., (PHENIX Collaboration). Phys. Rev. C **94**, 064901 (2016)
8. S. Acharya et al., (ALICE Collaboration). Phys. Lett. B **789**, 308–322 (2019)
9. A. Ayala, J.D. Castaño-Yepes, C.A. Dominguez, L.A. Hernandez, S. Hernandez-Ortiz, M.E. Tejeda-Yeomans, Phys. Rev. D **96**, 014023 (2017)
10. A. Ayala, J. D. Castaño-Yepes, I. Dominguez Jimenez, J. Salinas San Martín, M. E. Tejeda-Yeomans, Eur. Phys. J. A **56**, 53 (2020)
11. G. Basar, D.E. Kharzeev, V. Skokov, Phys. Rev. Lett. **109**, 202303 (2012)
12. G. Basar, D.E. Kharzeev, E.V. Shuryak, Phys. Rev. C **90**, 014905 (2014)
13. B.G. Zakharov, Eur. Phys. J. C **76**, 609 (2016)
14. K. Tuchin, Phys. Rev. C **91**, 014902 (2015)
15. J. Adam et al., (STAR Collaboration). Phys. Rev. C **98**, 014910 (2018)
16. Y. Guo, S. Shi, S. Feng, J. Liao, Phys. Lett. B **798**, 134929 (2019)
17. G.S. Bali, F. Bruckmann, G. Endrödi, Z. Fodor, S.D. Katz, S. Krieg, A. Schäfer, K.K. Szabó, J. High Energy Phys. **02**, 044 (2012)
18. G.S. Bali, F. Bruckmann, G. Endrödi, Z. Fodor, S.D. Katz, A. Schäfer, Phys. Rev. D **86**, 071502 (2012)
19. G. Bali, F. Bruckmann, G. Endrödi, S. Katz, A. Schäfer, J. High Energy Phys. **08**, 177 (2014)
20. F. Bruckmann, G. Endrödi, T.G. Kovacs, J. High Energy Phys. **04**, 112 (2013)
21. R.L.S. Farias, K.P. Gomes, G. Krein, M.B. Pinto, Phys. Rev. C **90**, 025203 (2014)
22. M. Ferreira, P. Costa, O. Lourenço, T. Frederico, C. Providência, Phys. Rev. D **89**, 116011 (2014)
23. A. Ayala, L.A. Hernández, A.J. Mizher, J.C. Rojas, C. Villavicencio, Phys. Rev. D **89**, 116017 (2014)
24. A. Ayala, M. Loewe, R. Zamora, Phys. Rev. D **91**, 016002 (2015)
25. A. Ayala, C.A. Dominguez, L.A. Hernández, M. Loewe, R. Zamora, Phys. Rev. D **92**, 096011 (2015)
26. A. Ayala, M. Loewe, A.J. Mizher, R. Zamora, Phys. Rev. D **90**, 036001 (2014)
27. R.L.S. Farias, V.S. Timoteo, S.S. Avancini, M.B. Pinto, G. Krein, Eur. Phys. J. A **53**, 101 (2017)
28. A. Ayala, C.A. Dominguez, L.A. Hernández, M. Loewe, A. Raya, J.C. Rojas, C. Villavicencio, Phys. Rev. D **94**, 054019 (2016)
29. A. Ayala, C.A. Dominguez, L.A. Hernández, M. Loewe, R. Zamora, Phys. Lett. B **759**, 99–103 (2016)
30. A. Ayala, J.J. Cobos-Martínez, M. Loewe, M.E. Tejeda-Yeomans, R. Zamora, Phys. Rev. D **91**, 016007 (2015)
31. N. Mueller, J.A. Bonnet, C.S. Fischer, Phys. Rev. D **89**, 094023 (2014)
32. N. Mueller, J.M. Pawlowski, Phys. Rev. D **91**, 116010 (2015)
33. G.S. Bali, B.B. Brandt, G. Endrödi, B. Gläßle, Phys. Rev. D **97**, 034505 (2018)
34. A. Ayala, R.L.S. Farias, S. Hernández-Ortiz, L.A. Hernández, D. Manreza-Paret, R. Zamora, Phys. Rev. D **98**, 114008 (2018)
35. A. Ayala, J.D. Castaño-Yepes, C.A. Dominguez, S. Hernández-Ortíz, L.A. Hernández, M. Loewe, D. Manreza-Paret, R. Zamora, Rev. Mex. Fis. **66**, 446–461 (2020)
36. M. Hasan, B.K. Patra, Phys. Rev. D **102**(3), 036020 (2020)
37. T.-K. Chyi, C.-W. Hwang, W.F. Kao, G.-L. Lin, K.-W. Ng, Phys. Rev. D **62**, 105014 (2000)
38. A. Ayala, A. Sanchez, G. Piccinelli, S. Sahu, Phys. Rev. D **71**, 023004 (2005)
39. A. Ayala, J.D. Castaño-Yepes, M. Loewe, E. Muñóz, Phys. Rev. D **101**, 036016 (2020)
40. V.A. Miransky, I.A. Shovkovy, Phys. Rept. **576**, 1–209 (2015)
41. V.E. Gusynin, V.A. Miransky, I.A. Shovkovy, Nucl. Phys. B **462**, 249–290 (1996)
42. K. Hattori, K. Itakura, Ann. Phys. **330**, 23–54 (2013)
43. B. Karmakar, A. Bandyopadhyay, N. Haque, M.G. Mustafa, Eur. Phys. J. C **79**(8), 658 (2019)