QUALITATIVE PROPERTIES OF NONLINEAR PARABOLIC EQUATIONS WITH DOMINATING GRADIENT TERMS

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A mi madre, Cristina, siempre

Un hombre se propone la tarea de dibujar el mundo. A lo largo de los años puebla un espacio con imágenes de provincias, de reinos, de montañas, de bahías, de naves, de islas, de peces, de habitaciones, de instrumentos, de caballos y de personas. Poco antes de morir, descubre que ese laberinto de líneas traza la imagen de su cara.

—Jorge Luis Borges, *El Hacedor*

What people who want to be writers need is to be put in an area that they cannot manuever out of by weak and dirty play.

—Charles Bukowski, *Upon the Mathematics of the Breadth and the Way*

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—Nicanor Parra, *Misión Cumplida*

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Contents

Chapter 1

Introduction

1.1 The viscous Hamilton-Jacobi equation

The present work is a contribution to the study of viscosity solutions of nonlinear parabolic equations. We address the phenomena of *continuous solvability*, *loss of boundary conditions* (defined below), and the *large-time behavior* for initial and boundary value problems associated to different generalizations of the so-called viscous Hamilton-Jacobi equation,

$$u_t - \Delta u = |Du|^p \text{ in } \Omega \times (0,T). \tag{1.1.1}$$

In Chapters 2 and 3, we study the analogues of (1.1.1) on a bounded, open $\Omega \subset \mathbb{R}^N$, when diffusion is given by a more general operator instead of the Laplacian. In Chapter 4, we study the large-time behavior of unbounded solutions of (1.1.1) with a nonhomogeneous source term when $\Omega = \mathbb{R}^N$.

For $p = 2$, (1.1.1) is a deterministic version of the celebrated Kardar-Parisi-Zhang equation. The latter was proposed by these authors in [41] as a model of a growing interface. Mathematically, it is of interest because it is a simple model of a parabolic equation with nonlinear dependence on the gradient, and also, because it results from applying the *vanishing viscosity* method to a first-order Hamilton-Jacobi equation (see e.g., [33], Ch. 10).

For equation (1.1.1), including all values $p > 0$, it is well-know that there exists a unique, maximal-in-time classical solution $u \in C^{1,\alpha}(\Omega \times [0,T^*])$ for some $\alpha > 0$ and $0 < T^* \leq \infty$, assuming sufficient regularity for Ω and for the initial and boundary data ([35], Ch. 7).

In [57] it is proved that globally defined, classical solutions of problem (1.1.1), subject to conditions

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.1.2}$$
$$u(\cdot,0) = u_0 \quad \text{in } \overline{\Omega}, \tag{1.1.3}$$

fail to exist if $p > 2$ and $u_0 \in C^1(\overline{\Omega})$ is suitably large. It is also shown therein that this implies the occurrence of *gradient blow-up* (GBU, for short). GBU is said to happen in finite time $0 < T < \infty$ if a solution u of (1.1.1) satisfies

$$\sup_{[0,T]\times\Omega} u < \infty, \quad \lim_{t\to T} \sup_{x\in\overline{\Omega}} |Du(x,t)| = \infty.$$

A version of equation (1.1.1) containing a more general gradient term with superquadratic growth is studied in [2] in the context of weak solutions, for irregular initial data. The notable result is the nonexistence of global-in-time weak solutions with initial data u_0 assumed to be a positive, bounded measure and suitably large.

In both [2] and [57], the largeness condition on u_0 is given in terms of an L^2-product with the principal eigenfunction of the Laplacian. The method of proof is sometimes commonly known as *the principal eigenfunction method.* The condition that appears in our Theorems 1.2.2 and 1.2.5 is essentially the same as the one in [57]. An alternative proof of global nonexistence for (1.1.1) given in [51], Theorem 40.2, uses a weaker condition on u_0: in this proof it is enough to consider its L^q-norm, for any $q \geq 1$, but the argument does not adapt to more general nonlinearities (see Sec. 2.5).

There are different extensions of the results of [57]. Still in the context of classical solutions, the existence of global solutions of equation (1.1.1) with nontrivial right-hand side is studied in [58], as well as their large-time (or asymptotic) behavior. For equation

$$u_t - \Delta u = |Du|^p + \lambda h(x) \quad \text{in } \Omega \times (0,T), \tag{1.1.4}$$

where $\lambda \geq 0$, $h \in C^1(\overline{\Omega})$, $h \geq 0$, a complete description of the asymptotic behavior is given when u_0, h are radially symmetric and $\Omega = B_R(0)$, for some $R > 0$: in this case, for $h \not\equiv 0$, there exists a $\lambda^* > 0$ such that

- if $0 \leq \lambda < \lambda^*$, then (1.1.4) has a global solution which converges to the solution of the steady-state equation

$$-\Delta v = |Dv|^p + \lambda h(x) \quad \text{in } \Omega, \tag{1.1.5}$$

  which additionally satisfies $v \in C^1(\overline{\Omega})$.

- if $\lambda = \lambda^*$, then u converges to a solution $v \notin C^1(\overline{\Omega})$ for any $u_0 \in C^1(\overline{\Omega})$ with $u_0 \leq v$. This implies GBU in *infinite time*, i.e.,

$$\limsup_{t\to\infty} \|Du(\cdot,t)\|_\infty = \infty.$$

- if $\lambda > \lambda^*$, then (1.1.5) has no solution and GBU in finite time occurs for *any* $u_0 \in C^1(\overline{\Omega})$.

In the case of a general, bounded domain $\Omega \subset \mathbb{R}^n$ only a partial description is given.

Some of the previous results have been extended to equations with degenerate diffusion (i.e., with Δ_p in place of Δ) in [5], in the context of weak solutions.

Other questions, such as determining precise blow-up rates, profiles and sets, are addressed in [61], [5], [45]. See also [51], Ch. IV, and the references therein.

Equation (1.1.1) has also been studied from the viewpoint of viscosity solutions, in which a generalized notion of boundary conditions exists. The relevant phenomenon in this context is known as *loss of boundary conditions* (LOBC, for short). More precisely, (1.1.2) is said to hold *in the viscosity sense* for a subsolution $u \in USC(\overline{\Omega})$ (resp., a supersolution $v \in LSC(\overline{\Omega})$) of (1.1.1) if

$$\min\left(u_t - \Delta u - |Du|^p, u\right) \le 0. \tag{1.1.6}$$

(resp.

$$\max\left(u_t - \Delta u - |Du|^p, u\right) \ge 0.) \tag{1.1.7}$$

Loss of boundary conditions is said to occur whenever (1.1.2) (alternatively, the equality involving u in (1.1.6) or (1.1.7)) is not satisfied in the classical sense. A standard reference for the concept of viscosity solutions is [25]; in particular see [25], Sec. 7, which covers generalized boundary conditions. Another helpful reference for this last topic can be found in [7], Chap. 4., where its relation to the underlying optimal control problem is covered.

In [10] it is proved that the problem (1.1.1)-(1.1.2)-(1.1.3) admits a unique, globally defined, continuous viscosity solution, assuming boundary conditions are understood in the generalized sense defined by (1.1.6) and (1.1.7). This follows from a strong comparison principle proved by these authors and a subsequent application of Perron's method. The result is relevant mainly in the case $p > 2$, since it is shown in this same work that for $p \le 2$ there is no LOBC for either sub- or supersolutions, hence the classical comparison result of [25] applies, and global existence of solutions satisfying Dirichlet boundary conditions in the classical sense follows. A one-dimensional example of LOBC is also provided in [10]. In contrast to those furnished by our results, this solution satisfies time-dependent boundary data. As the results of [10] apply directly to the problem considered in Chapter 2, we review some of them in Section 2.1 for convenience.

Building on the existence of global solutions of the generalized Dirichlet problem, a natural question is to determine their large-time behavior. In this direction again there is an important distinction between the sub- and superquadratic cases, which are studied rather thoroughly in [16] and [60], respectively. Consider equation

$$u_t - \Delta u + |Du|^p = f(x) \quad \text{ in } \Omega \times (0, T), \tag{1.1.8}$$

where $\Omega \subset \mathbb{R}^N$ is bounded,

$$u(x,t) = \varphi(x) \quad \text{on } \partial\Omega \times (0, T),$$

is satisfied *in the viscosity sense*, $f \in C(\overline{\Omega})$, $\varphi \in C(\partial\Omega)$, and $\varphi(x) = u_0(x)$ for all $x \in \partial\Omega$. In the superquadratic case, $p > 2$, there are two possibilities: if the corresponding steady-sate equation

$$-\Delta v + |Dv|^p = f(x) \quad \text{ in } \Omega \tag{1.1.9}$$

has a bounded subsolution, then there exists a solution u_∞ of (1.1.9) and $u(x,t) \to u_\infty$ on $\overline{\Omega}$. If (1.1.9) fails to have bounded subsolutions, one must introduce the so-called *ergodic problem* with state-constraint boundary conditions:

$$\begin{aligned} -\Delta v + |Dv|^p &= f(x) + c \quad \text{in } \Omega, \\ -\Delta v + |Dv|^p &\geq f(x) + c \quad \text{in } \partial\Omega. \end{aligned} \tag{1.1.10}$$

Here $c \in \mathbb{R}$ is the so-called *ergodic constant*, and is an unknown in problem (1.1.10) together with v. Existence and uniqueness of solutions (c, v) of (1.1.10) are studied in [42]: c is unique while v is unique up to an additive constant. Convergence of $u(x,t) + ct$ to v where (c, v) is a solution of (1.1.10), as well as LOBC is then analyzed.

The behavior in the subquadratic case is more complicated. It depends also on whether $1 < p \leq 3/2$ or $3/2 < p \leq 2$ and becomes necessary to introduce the following problem, also studied in [42], as an analogue of (1.1.9) and (1.1.10):

$$\begin{aligned} -\Delta v + |Dv|^p = f(x) + c \quad &\text{in } \Omega, \\ v(x) \to \infty \quad &\text{as } x \to \partial\Omega. \end{aligned} \tag{1.1.11}$$

We refer the reader to [16], which also contains a study of (1.1.10) and (1.1.11) in the context of viscosity solutions.

A different type of result concerning large-time behavior is given in [47]. It is shown that there exist constants K, λ and C such that the solution of the generalized Dirichlet problem for (1.1.8) with homogeneous boundary data, $f \equiv 0$, and any compatible initial data $u_0 \in C(\overline{\Omega})$ satisfies, for every $t \geq K\|u_0\|_\infty$,

$$u(\cdot, t) \in W^{1,\infty}(\Omega), \quad \text{and} \quad \|u(\cdot,t)\|_\infty + \|Du(\cdot,t)\|_\infty \leq Ce^{-\lambda t}.$$

In particular, this implies that after some finite time, the solution u satisfies the boundary data in the classical sense. This property is then applied to the interesting problem of the null controllability of (1.1.8).

Regarding regularity of solutions, it is proved in [23] that if u is a bounded, upper-semicontinuous viscosity subsolution of the (possibly degenerate) elliptic equation

$$-\operatorname{tr}(A(x)D^2u) + \lambda u + |Du|^p = f(x) \quad \text{for all } x \in \Omega, \tag{1.1.12}$$

where $p > 2$, $\Omega \subset \mathbb{R}^n$ is a regular domain, $\lambda > 0$, and $A : \Omega \to S(N)$ and f satisfy fairly standard assumptions, then u is globally Hölder continuous with exponent $\alpha = {p-2}/{p-1}$ (i.e., $u \in C^{0,{p-2}/{p-1}}(\overline{\Omega})$). As noted in [8], the result above is surprising, since most regularity results apply to actual solutions of uniformly elliptic equations that satisfy subquadratic growth conditions, none of which points are met in the assumed hypotheses. The authors of [23] go on to prove interior Lipschitz bounds for solutions of (1.1.12) by the so-called *weak-Bernstein* method introduced in [6]. These results are valid for fully-nonlinear equations satisfying hypotheses which are discussed in detail in [8].

In [8] a slight simplification of the proof of Hölder regularity is provided, and the relation to the solvability of Dirichlet problem is analyzed. In short, if a general boundary condition $u = \varphi$ with $\varphi \in C(\partial\Omega)$ is assumed in the viscosity sense, an additional reason for the occurrence of LOBC is that φ might not have the same regularity as u. This is, of course, irrelevant to the case of homogeneous boundary data.

Time-dependent versions of these regularity results are proven in [4], though they require the additional assumption that

$$u_t \geq -C \quad \text{for all } (x,t) \in \Omega \times (0,T)$$

for some $C \geq 0$ be satisfied *in the viscosity sense*. This means that: for all $(x,t) \in \Omega \times (0,T)$, if $(a,\xi,X) \in \mathcal{P}^{2,+}u(x,t)$, the parabolic superjet of a subsolution u (see, e.g., [25] for definitions), then $a \geq -C$. To the best of our knowledge, there is no readily available result in the context of viscosity solutions that would allow us to do away with this assumption, a common strategy being the smoothing of the data coupled with some sort of regularity result. Instead, we have followed the strategy of regularizing the solution itself (see Sec. 2.3).

1.2 Model problems and main results

The first of the problems considered in this thesis is in the fully nonlinear, second-order setting. Consider

$$u_t - \mathcal{M}^-(D^2u) = |Du|^p \quad \text{in } \Omega \times (0,T), \tag{1.2.1}$$

again subject to (1.1.2)-(1.1.3), where $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying both uniform interior and exterior sphere conditions. While this regularity for the domain is not strictly necessary for all our results, it does establish a better connection between our main theorems. See Remarks 2.2.2 and 2.5.4. For (1.2.1) (and throughout Chapter 2) we assume $p > 2$. The case $p \leq 2$ is addressed only for certain remarks made in this introduction. See also Remark 2.2.1. In (1.2.1), $\mathcal{M}^-$ denotes one of Pucci's extremal operators, which are defined as follows: let $A, X \in S(N)$, the symmetric $N \times N$ matrices equipped with the usual ordering, I denote the identity matrix, and $0 < \lambda < \Lambda$. Then

$$\mathcal{M}^-(X) = \mathcal{M}^-(X,\lambda,\Lambda) = \inf\{\operatorname{tr}(AX) \,|\, \lambda I \leq A \leq \Lambda I\},$$
$$\mathcal{M}^+(X) = \mathcal{M}^+(X,\lambda,\Lambda) = \sup\{\operatorname{tr}(AX) \,|\, \lambda I \leq A \leq \Lambda I\}.$$

Alternatively, if we denote by $\lambda_i = \lambda_i(X)$ the eigenvalues of X, then

$$\mathcal{M}^-(X) = \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i, \qquad \mathcal{M}^+(X) = \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i.$$

Pucci's operators are fundamental to the study of fully nonlinear equations, at once acting as barriers to all equations sharing the same ellipticity constants (owing to the first definition) and allowing fairly explicit computations to be carried out (owing to the second). The Dirichlet condition (1.1.2) will be considered both in the classical sense and in the generalized sense of viscosity solutions. We will stress the distinction when necessary. On the other hand, condition (1.1.3) is always meant in the classical (pointwise) sense. See Remark 2.1.5.

We assume the compatibility condition

$$u_0(x) = 0 \quad \text{for all } x \in \partial\Omega$$

is also satisfied in the pointwise sense and that $u_0 \in C^1(\overline{\Omega})$. As with the assumptions on Ω, it is not strictly necessary to assume this regularity for u_0 throughout, but helps establish a connection between our main theorems. Also, we assume $u_0 \geq 0$ without loss of generality, since (1.2.1) is invariant with respect to additive constants.

The main results concerning (1.2.1) are the following. We begin by proving the existence of solutions of (1.2.1)-(1.1.2)-(1.1.3) that for a small time satisfy the boundary data in the classical sense. The existence time depends only on a gradient bound for the initial data, the remaining constants usually considered universal.

Theorem 1.2.1. *Let $u_0 \in C^1(\overline{\Omega})$. There exists a $T^* > 0$, depending only on $\Lambda, \lambda, N, \Omega$ and $\|u_0\|_{C^1(\overline{\Omega})}$, such that the viscosity solution of* (1.2.1) *in $\Omega \times (0, T^*)$ satisfies* (1.1.2) *and* (1.1.3) *in the classical sense.*

Since we already have the existence result of [10], we need only show that (1.1.2) is satisfied is the classical sense. For this we use a barrier argument, following the construction of comparison functions used in [5] to show local existence of an equation with degenerate diffusion in the setting of weak solutions.

Next we prove the nonexistence of global solutions to the classical Dirichlet problem when $\Omega = B_1(0)$ and the initial data is radially symmetric and suitably large. Again, due to the global existence result of [10], this implies the occurrence of LOBC.

Theorem 1.2.2. *Let $u_0 \in C^1(\overline{B_1(0)})$ be a radial function. Then, there exist positive constants $\delta = \delta(\lambda, \Lambda, N)$ and $M = M(\lambda, \Lambda, N, p)$ such that, if*

$$\int_{\delta}^{1-\delta} u_0(r)\, dr > M \tag{1.2.2}$$

then the solution u of (1.2.1)-(1.1.2)-(1.1.3) *with $\Omega = B_1(0)$ and initial data u_0 has LOBC at some finite time $T = T(u_0)$.*

The proof of Theorem 1.2.2 uses key ideas from that of Theorem 2.1 in [57]. The main difficulty in adapting this proof is its crucial use of the divergence structure of the Laplacian by repeatedly using integration by parts. We remedy this problem by using the divergence form of the Pucci operator, available for radial solutions (see, e.g., [34]), and the regularization by inf-sup-convolution introduced in [43]. Combining these techniques we obtain an equation in divergence form which is satisfied point-wise and all of whose terms are integrable. Afterwards, the main complications are keeping track of the terms which depend on the regularization parameters and providing estimates which are independent of these. We also adapt a weighted, one-dimensional version of Poincaré's inequality, and make use of different results from [21] and [31] regarding the principal eigenvalue problem for the Pucci operator.

This result is extended to show LOBC occurs for solutions of (1.2.1) in a sufficiently regular bounded domain in Corollary 2.4.2, and then to equations with more general nonlinearities, first in the radially symmetric case, then also for a bounded domain as above. The most general result is the following. Consider

$$u_t - F(D^2u) = f(Du) \text{ in } \Omega \times (0,T), \tag{1.2.3}$$

where $F : S(N) \to \mathbb{R}$ is uniformly elliptic, i.e.,

$$\mathcal{M}^-(X-Y) \leq F(X) - F(Y) \leq \mathcal{M}^+(X-Y) \quad \text{for all } X, Y \in S(N), \tag{1.2.4}$$

and vanishes at zero, i.e., $F(0) = 0$, and $f : \mathbb{R}^N \to \mathbb{R}$ satisfies $f(\xi) \geq |\xi|^2 h(|\xi|)$ for all $\xi \in \mathbb{R}^N$, where $h : \mathbb{R} \to \mathbb{R}$ is positive, nondecreasing, grows more slowly than any positive power, and is such that $\xi \mapsto |\xi|^2 h(|\xi|)$ is convex. Precise hypotheses on h are given in Section 2.5.

Theorem 1.2.3. *Assume that F, f, and h are as described above. If additionally h satisfies*

$$\int_1^\infty \frac{1}{sh(s)}\, ds < \infty, \tag{1.2.5}$$

then there exists $u_0 \in C^1(\overline{\Omega})$, with $u_0 \geq 0$ and $u_0|_{\partial\Omega} = 0$, such that LOBC occurs for solutions of (1.2.3)-(1.1.2)-(1.1.3) *in some finite time $T = T(u_0)$.*

This result follows more or less easily from Theorem 1.2.2 and the main ideas used in its proof, as do the other extensions given in Section 2.5.

The second problem we address is a generalization of (1.1.1) featuring fractional diffusion. Consider

$$u_t + (-\Delta)^s u = |Du|^p \quad \text{in } \Omega \times (0,T), \tag{1.2.6}$$

now subject to

$$u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega \times (0,T) \tag{1.2.7}$$

in addition to (1.1.3). Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, $T > 0$, and $(-\Delta)^s$ denotes the well-known fractional Laplacian operator, defined as

$$(-\Delta)^s u(x,t) = C_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x,t) - u(y,t)}{|x-y|^{N+2s}}\, dy, \tag{1.2.8}$$

where $C_{N,s}$ is a normalization constant. See [29] for details. Note that, due to the nonlocal nature of this operator, in (1.2.7) we must prescribe values for the solution in the whole complement of the domain.

We assume $s \in (0,1)$, and impose the following restrictions on p when dealing with (1.2.6) (i.e., in Chap. 3),

$$s + 1 < p < \frac{s}{1-s}. \tag{1.2.9}$$

In particular, these imply that $s \in (0.618\ldots, 1)$, where $0.618\ldots$ is the constant sometimes called *reciprocal golden ratio*. We provide further comment on these restrictions below, after the statement of our main theorems. Moreover, we assume $u_0 \geq 0$ as before, and

$$u_0 \in C^\beta(\overline{\Omega}), \quad \text{where } \beta > \beta^* = \frac{p-2s}{p-1}, \tag{1.2.10}$$

together with the compatibility condition

$$u_0(x) = 0 \quad \text{for all } \partial\Omega. \tag{1.2.11}$$

As with (1.2.9), (1.2.10) might not be optimal, and is explained in the context of Theorem 1.2.4.

Under the structural assumptions of nondegeneracy of the diffusion and coercivity of the first order term (easily shown to be satisfied by (1.2.6) — see Remark 3.1.2), and the compatibility condition (1.2.11), together with the notion of boundary conditions *in the viscosity sense*, the existence of a unique solution of (1.2.6)-(1.2.7)-(1.1.3) defined globally in time, $u \in C(\overline{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,T))$ for all $T > 0$, is shown in [20], Theorem 7.1. This result is proved by means of a comparison result ([20], Theorem 3.2), and a subsequent application of Perron's method. See Sec. 3.1 for a precise definition of the notion of solution employed and further remarks on the application of the results of [20] to our problem.

We remark that there exist certain results concerning the regularity of solutions for problems related to (1.2.6)-(1.2.7)-(1.1.3) (see, e.g., [12], [14]). However, even if they were adapted to our setting, they do not provide the regularity needed for the proof of Theorem 1.2.5. For this reason we resort to a regularization procedure. See the remarks after the statement of Theorem 1.2.5.

Our study of (1.2.6) follows the same schema used in the fully nonlinear setting: the first result concerns local existence; i.e., the existence of solutions which, for small time, satisfy (1.2.7) in the classical sense.

Theorem 1.2.4. *Assume* (1.2.9) *and* (1.2.10). *Then, there exists a* $T^* > 0$, *depending only on* N, Ω, s, p *and* $[u_0]_{C^\beta(\overline{\Omega})}$, *such that the viscosity solution of* (1.2.6) *satisfies* (1.2.7) *and* (1.1.3) *in the classical sense for all* $0 \leq t \leq T^*$.

Due to the results of [20], it suffices to show that the globally defined viscosity solution of (1.2.6) satisfies (1.2.7) in the classical sense (see Sec. 3.1). This is accomplished by a barrier argument, i.e., the construction of a supersolution of (1.2.6)-(1.2.7)-(1.1.3) in a neighborhood of $\partial\Omega$. It is here that the restriction

(1.2.9) comes into play. Consider $s \in (0,1)$ fixed. The upper bound $p < \frac{s}{1-s}$ implies for the critical exponent in (1.2.10) that $\beta^* < s$, while the construction of the barrier ultimately relies on computing

$$F(\beta) = \int_{\mathbb{R}} \frac{(1+\beta)_+ + (1-\beta)_+ - 2}{|t|^{1+2\beta}}\, dt$$

for $\beta \in (0, 2s)$; more precisely, on the fact that $F(\beta) < 0$ for each $\beta \in (0, s)$ (see [27], [52] for details).

We note that neither the corresponding local existence result for (1.1.1) in [35] nor its extension to the fully nonlinear case in [49] require an upper bound for p (or, more generally, for the rate of growth of the gradient nonlinearity). In this sense our result might not be optimal. It is *stable*, however, in the sense that the restriction disappears as we approach the second-order case, since $\frac{s}{1-s} \to \infty$ as $s \to 1^-$.

The statement of our second result, concerning LOBC, involves the principal eigenfunction of the fractional Laplacian. We denote by (λ_1, φ_1) the solution pair for

$$\begin{cases} (-\Delta)^s \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2.12}$$

where the solution is normalized so that $\|\varphi_1\|_\infty = 1$. See Sec. 3.3 for details.

Theorem 1.2.5. *Assume* (1.2.9) *and* (1.2.10)*, and let* $T > 0$. *Then, there exists* $M > 0$*, depending only on* N, Ω, s, p*, and* T*, such that, if*

$$\int_\Omega u_0(x)\varphi_1(x)\, dx > M, \tag{1.2.13}$$

then the viscosity solution of (1.2.6)-(1.2.7)-(1.1.3) *has LOBC at some finite time prior to* T.

The proof of Theorem 1.2.5 also uses "principal eigenfunction method", as in Theorem 2.1 in [57]. In adapting this argument to the current setting, the main difficulty is the lack of regularity of solutions. More precisely, we would need solutions to satisfy the equation either in the weak sense, or in some pointwise sense; in the latter case, all terms in the equation must be summable. As mentioned earlier, the existing theory for our problem does not provide such regularity. We note that even in the case of (1.1.1) (for which LOBC is obtained in [46], among other results), where viscosity solutions are shown to be smooth, some approximation procedure is necessary to "integrate" the equation.

As before, we use regularization by inf-sup-convolutions. Afterwards, we require that various estimates related to (1.2.12) remain uniform with respect to the regularization parameters. In particular, we obtain the stability of solutions to (1.2.12) with respect to the varying domain (see Subsec. 3.3). For this part we also rely on fundamental estimates for the Dirichlet problem for the fractional Laplacian from [52].

The part of (1.2.9) that is relevant to Theorem 1.2.5 is $s+1<p$. This assumption appears only in the crucial Lemma 3.3.14, which states that a certain negative power (given by p and s) of the principal eigenfunction on an approximate domain is summable. The restriction (1.2.9) is thus related to our method of proof. However, a lower bound for p in terms of s is necessary for LOBC to occurr: it is known that solutions of (1.2.6) satisfying (1.2.7) in the classical sense for all $T>0$ exist for $s\in[0,1]$ and $p\leq 2s$ (see [9], Theorem 4, for the nonlocal case and [10] for the local case, i.e., $s=1$). A natural question which we leave open is whether there is classical solvability or if LOBC occurs when $s\in[0,1)$ and $2s<p\leq s+1$.

For simplicity, we have restricted our analysis to the case of homogeneous boundary conditions, as in (1.2.7). Local existence for more general boundary conditions can be obtained in the same way as in Theorem 1.2.4, following the construction of [27]. Theorem 1.2.5 applies to the case of general boundary conditions with practically no modification (see Remark 3.4.1).

The methods of Theorem 1.2.5 apply to more general nonlinear operators as well, provided they satisfy the nonlocal equivalent of having divergence form (see [56], Sec. 3.6). This restriction is due to the essential use of "integration by parts" in the so-called principal eigenfunction method. For instance, the result can be extended to an equation with diffusion given by the so-called p-fractional Laplacian, defined for $s\in(0,1)$, $p>1$ and $x\in\mathbb{R}^N$ as

$$(-\Delta_p)^s u(x) = C(N,s,p)\,P.V.\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy. \quad (1.2.14)$$

In this case the technical results of Sec. 3.3 can be reproduced following [39] and [28].

Finally, in Chapter 4 we address the problem of large-time behavior of the solution $u=u(x,t)$ of

$$u_t-\Delta u+|Du|^m = f(x) \quad \text{in } \mathbb{R}^N\times(0,\infty), \quad (1.2.15)$$

$$u(\cdot,0)=u_0 \quad \text{in } \mathbb{R}^N, \quad (1.2.16)$$

where, $m>2$, $f\in W^{1,\infty}_{loc}(\mathbb{R}^N)$ is, at least, bounded from below. Additional assumptions on f are stated for each result, when necessary, since they differ for, e.g., the respective results on the existence and uniqueness of solutions. The initial data $u_0\in C(\mathbb{R}^N)$ is again assumed nonnegative, but in this case, this is equivalent to assuming u_0 is bounded from below.

There is of course a vast literature regarding large-time behavior for first- and second-order Hamilton-Jacobi equations. In particular, our work can be seen as a continuation of [60], [16], already cited in this Introduction. The main difficulty in our study is the fact that the domain is unbounded, and that under our assumptions for f and u_0, the solutions of (1.2.15)-(1.2.16) may become unbounded as well. Our first result is the following:

Theorem 1.2.6. *Assume $f\in W^{1,\infty}_{loc}(\mathbb{R}^N)$ and $u_0\in C(\mathbb{R}^N)$ are nonnegative. Then, there exists a unique, nonnegative, continuous solution of* (1.2.15)-(1.2.16).

The proof of Theorem 1.2.6 follows from applying Perron's method (see e.g. [25]) together with a comparison result, Theorem 4.1.4.

For first-order equations, the problem of unbounded solutions over the whole space is studied in [18], [19], [13], among others, but to the best of our knowledge there are no comparable results in the second-order case. The expected behavior for solutions of solutions to (1.2.15)-(1.2.16) is locally uniform convergence to the solution of the ergodic problem,

$$\lambda - \Delta\phi + |D\phi|^m = f(x) \quad \text{in } \mathbb{R}^N, \tag{1.2.17}$$

where both λ and ϕ are unknown. Assuming that $m > 2$ and $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ is coercive and bounded from below, it is proved in [15] that there exist unique $\lambda^* \in \mathbb{R}$ and $\phi \in C^2(\mathbb{R}^N)$ which together solve (1.2.17) (to be precise, ϕ is unique up to an additive constant). See Theorems 2.4 and 3.1 therein. In addition to using a number of results of [15] concerning the solution ϕ, we use key ideas and computations contained in the proof of Proposition 3.2 for the proof of our Theorem 4.1.4.

To present our result on the large-time behavior of solutions to (1.2.15)-(1.2.16), we first state the precise hypotheses on the data:

(H1) There exist $f_0, \alpha > 0$ such that, for all $x \in \mathbb{R}^N$

$$|Df(x)| \le f_0(1 + |x|^{\alpha-1}), \text{ if } \alpha \ge 1, \quad \text{ or } \quad |Df(x)| \le f_0, \text{ if } \alpha < 1.$$

(H2) There exist $f_0, \alpha > 0$ such that, for all $x \in \mathbb{R}^N$

$$f_0^{-1}(|x|^\alpha + 1) \le f(x).$$

(H3) The initial data in (1.2.16) satisfies $u_0 \in C(\mathbb{R}^N)$.

Theorem 1.2.7. *Under assumptions (H1), (H2) and (H3), there exists $\hat{c} \in \mathbb{R}$ such that*

$$u(x,t) - \lambda^* t \to \phi(x) + \hat{c} \quad \textit{locally uniformly in } \mathbb{R}^N \textit{ as } t \to \infty.$$

The proof of Theorem 1.2.7 follows the strategy and uses key ideas from the corresponding result on bounded domains, Theorem 4.1 (ii) in [60]. The main difficulty of the proof is having uniform convergence only locally in the places where we would wish to have uniform convergence or, alternatively, the lack of control of the solution on the boundary of bounded domains. This is remedied by the special sub- and super solutions of Lemma 4.2.1, which provide the required control "at infinity", i.e., in the complement of compact sets.

The hypotheses (H1) and (H2) are crucial to the construction of said sub- and supersolutions. These also imply that the solution of the ergodic problem (1.2.17) has superlinear growth at infinity, the use of which is also an important feature in the proof of Theorem 1.2.7. The assumption (H3) on the other hand, might not be essential. For the corresponding result on a bounded domain $\Omega \subset \mathbb{R}^N$, it is relatively simple to prove the general case of $u_0 \in C(\overline{\Omega})$

by an approximation argument. However, since, e.g., smoothing by standard convolution provides only local uniform convergence to continuous functions on $\mathbb{R}^N$, such an argument is not immediate adaptable to our setting, even with the techniques used to prove Theorem 1.2.7.

1.3 Notation and organization

Except for Definition 2.3.4 and Proposition 2.3.5, which are used in Chapters 2 and 3, the Chapters are self-contained (in particular, with regard to notation). The contents of each Section are outlined at the beginning of each Chapter.

Most of the notation appearing in the text is standard. In Chapter 3, we write $d : \mathbb{R}^N \to \mathbb{R}$, $d = d(x)$ for the *distance to the boundary* of the set Ω, extended by zero to the whole of $\mathbb{R}^N$, i.e.,

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \overline{\Omega}, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash \overline{\Omega}. \end{cases} \tag{1.3.1}$$

For $\delta > 0$, we write $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$, where $d = d(x)$ is defined as above. Similarly, $\Omega^\delta = \{x \in \Omega : d(x) > \delta\}$. To avoid confusion, we abstain from using both notations in the same section: in Sec. 3.2 we use the notation Ω_δ, and from Sec. 3.3 onwards we use only Ω^δ. The closure and boundary operation on sets is performed "after" specifying a subset in terms of the distance: e.g., $\overline{\Omega}_\delta = \{x \in \Omega : d(x) \leq \delta\}$. In Sec. 3.3 and Appendix 3.A we write, for $\eta > 0$, $d_\eta = d_\eta(x)$ for the distance to the boundary of Ω^η, extended by zero outside this set, as in (1.3.1).

In Chapter 4 we use the notation $\mathbb{R} \cup \{+\infty\}$ for the extended real numbers and, for an extended-real valued function $v : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$, $\operatorname{dom}(v) = \{x \in \mathbb{R}^N \mid v(x) < +\infty\}$.

Nonnegative constants whose precise value does not affect the argument are denoted collectively by C, and the value of C may change from line to line. When convenient, dependence of C on certain parameters is indicated in parentheses, e.g., as in (1.2.14). Dependency on, e.g., Ω, N, p, s, is sometimes omitted for simplicity. Constants we wish to keep track of are numbered accordingly ($c_1, c_2, \ldots, C_0, C_1$, etc.)

Chapter 2

Loss of boundary conditions for fully nonlinear parabolic equations with superquadratic gradient terms[1]

The Chapter is organized as follows. In Section 2.1 we briefly review the results of [10] which are directly used in our work. Section 2.2 is devoted to the proof of Theorem 1.2.1. In Section 2.3 we gather the technical results which lead us to the approximate equation we use to prove the nonexistence result, as well as some fundamental facts and estimates related to the eigenvalue problem for the Pucci extremal operator in a radial case. The statements and remarks of this section contain key concepts and notation used in the proof of the Theorem 1.2.2 and its subsequent generalizations. Section 2.4 contains the proof of our main result in the radially symmetric case, Theorem 1.2.2, and its extension to a bounded domain. This is the core of our work. Finally, in Section 2.5 we provide extensions to more general equations, including the proof of Theorem 1.2.3.

2.1 Comparison, existence and uniqueness

Existence and uniqueness for the so-called generalized Dirichlet problem for

$$u_t + G(x,t,Du,D^2u) = 0 \quad \text{in } \Omega \times (0,T), \tag{2.1.1}$$

where the boundary condition

$$u = g \quad \text{on } \partial\Omega \times (0,T), \tag{2.1.2}$$

$g \in C(\partial\Omega \times (0,T))$, is understood in the viscosity sense, is proven in Theorem 5.1 in [10]. For convenience, in this section we quote the main results of this

[1] This chapter is based on the article [49].

work, as well as a couple of remarks relevant to our purposes. Here G is a continuous function that satisfies the degenerate ellipticity condition,

$$G(x,t,\xi,X) \le G(x,t,\xi,Y) \quad \text{if } X \ge Y, \tag{2.1.3}$$

for all $x \in \overline{\Omega}$, $t \in [0,T]$, $\xi \in \mathbb{R}^n$ and $X, Y \in S(N)$, together with two key hypothesis, for which we must introduce additional notation. Note that condition (2.1.3) uses the opposite sign convention than the one used in (1.2.4). See also the discussion at the beginning of Subsection 2.5.

Let $h_1 : [0,\infty) \to [0,\infty)$ be a continuous function. We say h_1 satisfies property (P) if the following hold:

(i) $\int_1^\infty \frac{s}{h_1(s)}\, ds < \infty$,

(ii) for any $C > 0$, s large enough and $L \ge 1$, the map $L \mapsto h_1(Ls) - CL^2 h_1(s)$ is increasing,

(iii) for any $C, \tilde{C} > 0$, there exists $\bar{s} > 0, \bar{L} \ge 1$ such that

$$h_1(Ls) - CL^2 h_1(s) \ge \tilde{C} Ls \quad \text{for } s \ge \bar{s}, L \ge \bar{L}. \tag{2.1.4}$$

The key assumptions on G as the following:

(H1) There exists constants $C_1, C_2 > 0$ and a continuous function h_1 satisfying property (P) such that, for all $x \in \overline{\Omega}$, $t \in [0,T]$, $\xi \in \mathbb{R}^n$ and $X \in S(n)$, we have

$$G(x,t,\xi,X) \ge -C_1 - C_2\|X\| + h_1(|\xi|). \tag{2.1.5}$$

(H2) For any $\epsilon > 0$, there exists $0 < \mu_\epsilon < 1$ converging to 1 as $\epsilon \to 0$ such that

$$G(y,s,\xi_2,Y) - G(x,t,\mu_\epsilon^{-1}\xi_1, \mu_\epsilon^{-1}X) \le o(1)$$

for all $x, y \in \overline{\Omega}, t, s, \in [0,T], \xi_1, \xi_2 \in \mathbb{R}^n$ and for all $X, Y \in S(N)$ satisfying the following properties for some $K > 0$ and a sufficiently small $\eta > 0$:

$$-\frac{K\eta}{\epsilon^2} I_{2n} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{o(1)}{\epsilon^2} \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} + o(1) I_{2N},$$
$$|\xi_1 - \xi_2| \le K\epsilon \min\{|\xi_1|, |\xi_2|\},$$
$$|x-y| + |t-s| < \epsilon.$$

The main result is the following:

Theorem 2.1.1 (Strong Comparison Result)**.** *Assume $u_0 \in C(\overline{\Omega})$, and let u and v be respectively a bounded upper-semicontinuous (USC, for short) supersolution and a bounded lower-semicontinuous (LSC) supersolution of* (2.1.1)-(2.1.2)-(1.1.3)*, where G satisfies hypotheses* (H1) *and* (H2)*. Then $u \le v$ in $\Omega \times [0,T]$. Moreover, if we define $\tilde{u}$ on $\overline{\Omega} \times [0,T]$ by setting*

$$\tilde{u}(x,t) = \begin{cases} \limsup\limits_{\substack{(y,s)\to(x,t) \\ (y,s)\in\Omega\times(0,T)}} u(y,s) & \text{on } \partial\Omega \times (0,T], \\ u(x,t) & \text{otherwise,} \end{cases} \tag{2.1.6}$$

and similarly define $\tilde{v}$, then $\tilde{u}$ and $\tilde{v}$ are still respectively a bounded USC subsolution and a bounded LSC supersolution of (2.1.1)-(2.1.2)-(1.1.3) *and* $\tilde{u} \le \tilde{v}$ *in* $\overline{\Omega} \times [0, T]$.

As is standard, existence is proven by combining this result with Perron's method of sub- and supersolutions.

Remark 2.1.2. When Theorem 2.1.1 is used to compare *continuous* sub- and supersolutions, comparison holds up to the boundary without having to redefine the functions as in (2.1.6).

Remark 2.1.3. The lower bound of ((H1)) implies that the gradient nonlinearity has the opposite sign to that of (1.2.1). However, the results proved for

$$u_t - \mathcal{M}^+(D^2 u) + |Du|^p = 0 \quad \text{ in } \partial\Omega \times (0, T)$$

are valid for (1.2.1) provided we exchange the role of sub- and supersolutions. Indeed, u is a subsolution of the above equation if and only if $-u$ is a supersolution of (1.2.1). This is already noted in Remark 3.2 of [10]. We note also that in [10] there is no requirement that the solution be nonnegative, as there is in the proofs of gradient blow-up given in [57].

We will verify that hypotheses (H1) and (H2) apply to the equations considered in this work (after the appropriate sign change) in Section 2.5.

Remark 2.1.4. Following the exchange of sub- and supersolutions mentioned in the previous remark, it follows from Proposition 3.1 in [10] that any supersolution v of (1.2.1) satisfies $v \ge 0$ on $\partial\Omega \times (0, T)$ in the classical sense for any given $T > 0$. Hence, if LOBC occurs, as we prove later, then the solution satisfying (1.1.2) in the generalized sense must become strictly positive at some point of the boundary.

Remark 2.1.5. As mentioned in the introduction, the initial condition (1.1.3) is always meant in the classical sense. There is no loss of generality in this assumption. It is a consequence of Lemma 4.1 in [44] that there is no LOBC on the bottom of the parabolic domain, $\overline{\Omega} \times \{t = 0\}$.

Remark 2.1.6. An easy but important consequence of the comparison result is that solutions u of (1.2.1)-(1.1.3) are uniformly bounded and nonnegative. Indeed, for $u_0 \ge 0$, $\underline{v} \equiv 0$ and $\bar{v} \equiv \sup_{\overline{\Omega}} u_0$ are respectively sub- and supersolutions, so by comparison we have

$$0 \le u(x, t) \le \sup_{\overline{\Omega}} u_0 \quad \text{for all } x \in \overline{\Omega}, 0 \le t \le T. \tag{2.1.7}$$

In particular, $\|u\|_\infty \le \|u_0\|_\infty$.

2.2 Existence of local solutions

We follow the construction of the comparison functions used to prove local existence of solutions for a related problem in [5], accounting for the presence of the extremal operators and providing additional detail regarding the choice of constants.

Proof of Theorem 1.2.1: Step 1: A time-independent barrier. We define a time-independent comparison function in a neighborhood of a fixed $x_0 \in \partial\Omega$, using the exterior sphere condition, and prove that it is a supersolution of (1.2.1). We will address the initial and boundary conditions in a later step.

From the exterior sphere condition there exists a ball of radius $\rho > 0$ centered at $x_1 \notin \overline{\Omega}$, tangent to $\partial\Omega$ at x_0. We will employ the radial variables $r = |x - x_1|$, where $\rho < r < \rho + \eta$ for some $\eta > 0$, and $s = |x - x_1| - \rho$. In this and the following steps we will compare the solution u to different functions in the set

$$\Gamma = \{x \in \Omega \mid 0 < s = |x - x_1| - \rho < \eta\}.$$

Let $\varphi(s) = s(s+\mu)^{-\beta}$ with $\mu, \beta > 0$ to be chosen later, and define

$$\bar{v}(x) = \varphi(|x - x_1| - \rho) = \varphi(s). \tag{2.2.1}$$

For any C^2 radial function, say $\phi(x) = \phi(|x|)$, a standard computation of the eigenvalues of $D^2\phi$ at any point gives them explicitly as ϕ'' and ${}^{\phi'}/_{|x|}$ with multiplicities 1 and $N-1$, respectively, where $'$ denotes the derivative in the radial direction. By the definition of the extremal operator, this gives

$$\mathcal{M}^-(D^2\phi) = \min_{a,b \in \{\lambda,\Lambda\}} (a\phi'' + b\frac{N-1}{r}\phi'). \tag{2.2.2}$$

Setting $\beta < 1$, we compute

$$\begin{aligned}
\varphi'(s) &= [(1-\beta)s + \mu](s+\mu)^{-\beta-1} > 0,\\
\varphi''(s) &= -\beta[(1-\beta)s + 2\mu](s+\mu)^{-\beta-2} < 0.
\end{aligned}$$

Hence, the extremal operator takes the form

$$\begin{aligned}
\mathcal{M}^-(D^2\bar{v})(s) &= \Lambda\varphi''(s) + \lambda\left(\frac{N-1}{s+\rho}\right)\varphi'(s)\\
&= -\Lambda\beta[(1-\beta)s + 2\mu](s+\mu)^{-\beta-2}\\
&\quad + \lambda\left(\frac{N-1}{s+\rho}\right)[(1-\beta)s+\mu](s+\mu)^{-\beta-1}.
\end{aligned}$$

The function $\bar{v}$ is a supersolution if

$$-\mathcal{M}^-(D^2\bar{v}) \geq |\nabla\bar{v}|^p = |\varphi'|^p.$$

That is, from the previous computations, if

$$\begin{aligned}
&\left(\Lambda\beta[(1-\beta)s + 2\mu] - \lambda\left(\frac{N-1}{s+\rho}\right)[(1-\beta)s+\mu](s+\mu)\right)(s+\mu)^{-\beta-2}\\
&\qquad\qquad \geq [(1-\beta)s+\mu]^p(s+\mu)^{-p(\beta+1)}.
\end{aligned}$$

Here we have factored the leading term $(s+\mu)^{-\beta-2}$ in the left-hand side. We proceed to show that its coefficient K is positive for the right choices of μ and β.

Setting $\eta = \mu$ and using only that $0 < \beta < 1$ and $0 < s < \eta = \mu$, we have

$$K > 2\Lambda\beta\mu - \lambda\left(\frac{N-1}{s+\rho}\right)((1-\beta)\mu + \mu)2\mu.$$

Hence, to have $K > 0$ it is sufficient that

$$\mu < \frac{\beta}{2-\beta}\left(\frac{2\Lambda\rho}{\lambda(N-1)}\right). \tag{2.2.3}$$

Next, we verify that

$$K(s+\mu)^{-\beta-2} \geq [(1-\beta)s+\mu]^p (s+\mu)^{-p(\beta+1)}. \tag{2.2.4}$$

Again $0 < \beta < 1$ implies

$$[(1-\beta)s+\mu]^p \leq (s+\mu)^p,$$

then

$$\begin{aligned}[(1-\beta)s+\mu]^p (s+\mu)^{-p(\beta+1)} &\leq (s+\mu)^p (s+\mu)^{-p-p\beta} \\ &= (s+\mu)^{-p\beta}.\end{aligned}$$

Hence (2.2.4) holds if $(s+\mu)^{-p\beta} \leq K(s+\mu)^{-\beta-2}$, that is, if

$$K^{-1} \leq (s+\mu)^{\beta(p-1)-2}. \tag{2.2.5}$$

Setting $\beta < \frac{1}{2(p-1)}$ gives $\beta(p-1)-2 < -\frac{3}{2}$, so that the term on the right is singular. This precise value of β will be useful in a moment. Using once more that $0 < s < \mu$, it is sufficient to have

$$K^{-1} \leq (2\mu)^{\beta(p-1)-2}. \tag{2.2.6}$$

We recall that K, the coefficient defined above, also depends on μ. However, from the above computations we have that for small μ,

$$K \geq 2\Lambda\beta\mu - \lambda\left(\frac{N-1}{s+\rho}\right)((1-\beta)\mu+\mu)2\mu \geq C_1\mu - C_2\mu^2,$$

hence $K^{-1} = O(\mu^{-1})$ as $\mu \to 0$, whereas the previous choice for β gives that the right-hand side of (2.2.6) is $O(\mu^{-\frac{3}{2}})$. Therefore, choosing μ small enough gives all the desired inequalities.

Step 2: Time-dependent control. We introduce a second comparison function which will help us relate the solution u of (1.2.1)-(1.1.2)-(1.1.3) to the supersolution $\bar{v}$ constructed in the previous step.

Let

$$\bar{u}(x,t) = At + C(1 - e^{-\gamma s}),$$

and write $\psi(s) = 1 - e^{-\gamma s}$. We will prove that for appropriate choices of the positive constants A, C and γ, $\bar{u}$ satisfies

$$\bar{u}_t - \mathcal{M}^-(D^2\bar{u}) \geq |D\bar{u}|^p \quad \text{in } \Gamma \times (0,\infty), \tag{2.2.7}$$

$$\bar{u} \geq u \quad \text{on } (\partial\Gamma \cap \Omega) \times (0,\infty), \tag{2.2.8}$$

$$\bar{u} \geq u_0 \quad \text{on } \overline{\Gamma} \times \{t = 0\}, \tag{2.2.9}$$

where (2.2.8) and (2.2.9) are meant in the classical sense.

Denote by ν the exterior unit normal at $x_0 \in \partial\Omega$. For $x = x_0 - s\nu,\ t = 0$, this is

$$\bar{u}(x,0) = C(1 - e^{-\gamma s}) \geq u_0(x).$$

We use that

$$0 < \left|\frac{\partial \bar{u}}{\partial \nu}(x_0)\right| = C\psi'(0) = C\gamma < \infty, \tag{2.2.10}$$

$\|Du_0\|_\infty < \infty$, $u_0(x_0) = \psi(0) = 0$, and that both u_0 and $\bar{u}$ are non-negative to choose $C > 0$ large enough, so that for small s, say $0 < s < \delta$, we have

$$u_0(x) = u_0(x_0 - s\nu) \leq C\psi(s).$$

In other words, we are comparing the first-order expansions in the direction $-\nu$. Then, for $\delta \leq s \leq \eta$, we may also take

$$C \min\{1, \min_{\delta \leq s \leq \eta} \bar{u}(x_0 - s\nu, 0)\} > \max_{\Omega \times [0,T]} u_0, \tag{2.2.11}$$

since the minimum above is strictly positive. We may repeat this reasoning in the other directions which sweep $\overline{\Gamma}$, by considering an extension by zero of u_0 to the corresponding section of the annular domain where ψ is defined. The choice of C remains bounded since $\overline{\Gamma}$ is compact. Furthermore, it can be chosen uniformly with respect to x_0 since $\overline{\Omega}$ is compact. Observe that this also ensures that $\bar{u} \geq u$ on the rest of $\partial_p(\Gamma \times (0,\infty))$.

To check (2.2.7), we compute

$$-\mathcal{M}^-(D^2\bar{u}) = \left(\Lambda\gamma - \lambda\frac{N-1}{s+\rho}\right) C\gamma e^{-\gamma s}, \tag{2.2.12}$$

and observe that choosing γ large enough gives $-\mathcal{M}^-(D^2\bar{u}) \geq 0$. Then, $\bar{u}$ is a supersolution if we can get

$$\bar{u}_t \geq |D\bar{u}|^p.$$

This amounts to taking

$$A \geq \max_{0 \leq s \leq \eta} (C\gamma e^{-\gamma s})^p.$$

We have therefore proved that $\bar{u}$ satisfies (2.2.7)-(2.2.8)-(2.2.9). Also, by definition $\bar{u} \geq 0$ on $\{x_0\} = \partial\Gamma \cap \partial\Omega$. Hence, by comparison, it follows that $\bar{u} \geq u$ in all of $\Gamma \times [0,\infty)$.

Step 3. Relating the comparison functions for small time. We claim that for some $T^* > 0$,

$$\bar{u}(x,t) \leq \bar{v}(x) \quad \text{for all } x \in \Gamma \text{ and } 0 \leq t \leq T^*.$$

The proof is similar to that of the previous step. We establish first the comparison for $t = 0$, the bottom of the domain. Again we consider $x = x_0 - s\nu$. Recalling (2.2.10), we now seek

$$\left|\frac{\partial \bar{v}}{\partial \nu}(x_0)\right| = \varphi'(0) \geq C\gamma = \left|\frac{\partial \bar{u}}{\partial \nu}(x_0, 0)\right|. \tag{2.2.13}$$

From previous computations,

$$\varphi'(0) = \mu^{-\beta} \to +\infty \quad \text{as } \mu \to 0.$$

On the other hand, C depends on μ through (2.2.11). Since $\psi = \psi(s)$ is increasing inward, the minimum in (2.2.11) is achieved at $s = \delta$. Clearly we can take $\delta < \mu = \eta$, and so a simple computation shows

$$C = o(\mu^{-\beta}) \quad \text{as } \mu \to 0.$$

Therefore, taking $\mu = \eta$ small enough eventually yields (2.2.13). We remark that this choice, which amounts to shrinking the domain Γ, does not affect the choices made for other constants.

As before, looking at the first order expansion gives $\bar{u}(x,0) \leq \bar{v}(x)$ for all x as above, near x_0, say with $s < \delta'$. Moreover, in this case it is easier to extend the inequality to the directions which sweep $\Gamma \times \{t = 0\}$, since both functions are radial and defined on the same annular domain.

To obtain

$$\bar{u}(x,0) \leq \bar{v}(x) \quad \text{on } \partial\Gamma, \tag{2.2.14}$$

there is no choice like (2.2.11) available. However, we may restrict the comparison to $\Gamma_{\delta'} \times \{t = 0\}$, where $\Gamma_{\delta'} := \Gamma \cap \{0 < s = |x - x_1| - \rho < \delta'\}$, $\delta' > 0$ as above, so that (2.2.14) holds by comparing the first-order expansions, and more importantly, holds strictly. That is, $\bar{u}(x,0) < \bar{v}(x)$. We may now take T^* small enough so that, for all $0 \leq t \leq T^*$ and all $x \in \Omega$ such that $s = |x - x_1| - \rho = \delta'$,

$$u(x,t) \leq \bar{u}(x,t) = At + \bar{u}(x,0) \leq \bar{v}(x).$$

Thus we have proven that $\bar{v}$ solves

$$\begin{cases} \bar{v}_t - \mathcal{M}^-(D^2\bar{v}) \geq |D\bar{v}|^p & \text{in } \Gamma_{\delta'} \times (0, T^*), \\ \bar{v} \geq u & \text{on } (\partial\Gamma_{\delta'} \cap \overline{\Omega}) \times [0, T^*], \\ \bar{v} \geq 0 & \text{on } (\partial\Gamma_{\delta'} \cap \partial\Omega) \times [0, T^*], \\ \bar{v} \geq u_0 & \text{in } \overline{\Gamma}_{\delta'}, \end{cases}$$

where the boundary conditions (inequalities) are satisfied pointwise. Hence, by the comparison principle of [10] (see also Remark 2.1.2), $u(x,t) \leq \bar{v}(x)$ in all of $\overline{\Gamma}_{\delta'} \times [0,T^*]$. In particular, this implies $u(x_0,t) \leq \bar{v}(x_0) = 0$, hence $u(x_0,t) = 0$ for all $0 \leq t \leq T^*$. As $x_0 \in \partial\Omega$ was arbitrary, this implies that the solution u satisfies the boundary conditions in the classical sense on $\partial\Omega \times [0,T^*]$. $\square$

Remark 2.2.1. The preceding computations do not require that $p > 2$; only $p > 1$ is explicitly used in (2.2.4). We remark, however, that only the superquadratic case is of interest, since by the results of [10], in the subquadratic case the globally defined solution of (1.2.1)-(1.1.2)-(1.1.3) satisfies the boundary data in the classical sense.

Remark 2.2.2. The proof of Theorem 1.2.1 uses only the uniform exterior sphere condition. Both interior and exterior sphere conditions are assumed to establish a connection between Theorems 1.2.1 and 1.2.3. See also Remark 2.5.4.

2.3 Technical results for the proof of nonexistence

In this section we gather a series of technical results and fundamental facts related, on one hand, to the process by which we arrive at an approximate equation (actually, an inequality) with the required properties, and on the other, to the eigenvalue problem for the Pucci operator. Furthermore, we introduce key concepts and notation that will be used in Sec. 2.4.

Radial form

Lemma 2.3.1. *Let $u \in C(\overline{B_1(0)})$ be the viscosity solution of*

$$\begin{cases} u_t - \mathcal{M}^-(D^2u) - |Du|^p = 0 & \text{in } B_1(0)\times(0,T), \\ u = 0 & \text{on } \partial B_1(0)\times[0,T], \\ u(\cdot,0) = u_0 & \text{in } \overline{B_1(0)}, \end{cases} \tag{2.3.1}$$

where u_0 is a radial function. Then u is radial as well, that is, $u(x,t) = U(|x|,t)$ for some $U : [0,1]\times[0,T] \to \mathbb{R}$, and U solves

$$\begin{cases} U_t - \theta(U'')U'' - \frac{N-1}{r}\theta(U')U' - |U'|^p = 0 & \text{in } (0,1)\times(0,T), \\ U = 0 & \text{on } \{r=1\}\times[0,T], \\ U(\cdot,0) = u_0 & \text{in } B_1(0), \end{cases} \tag{2.3.2}$$

in the viscosity sense, where $'$ denotes the radial derivative and

$$\theta(s) = \begin{cases} \lambda, & \text{if } s > 0, \\ \Lambda, & \text{if } s \leq 0. \end{cases}$$

Remark 2.3.2. We note that, although the function θ above is discontinuous at 0, equation (2.3.2) depends continuously on the derivatives of u, since the function $s \mapsto \theta(s)s$ is continuous for all $s \in \mathbb{R}$.

Proof: The solution u of (2.3.1) is radial due to the uniqueness of solutions of the Cauchy-Dirichlet problem, the rotation-invariance of the equation and the fact that the initial data is radial. Hence, there exists a function $U : \mathbb{R} \to \mathbb{R}$ such that $u(x) = U(|x|)$ for all $x \in B_1(0)$. We will show that this function is a subsolution of (2.3.2) by definition.

Consider $\Phi((0,1) \times (0,T)) \in C^2$ that touches U from above at $(\hat{r}, \hat{t})$, and define $\phi(x,t) = \Phi(|x|, t)$. Then ϕ is C^2, radial and a valid test function for u at any $(\hat{x}, \hat{t})$ such that $|\hat{x}| = \hat{r}$, hence we can compute $\mathcal{M}^-(D^2\phi)$ as in (2.2.2). Observing also that $|D\phi| = |\Phi'|$, we obtain exactly the equation in (2.3.2). The proof that U is also a supersolution is analogous. □

Remark 2.3.3. In what follows we will at times write simply $u(x) = u(r)$ for radial functions, as is standard. We avoided this notation in the last lemma for clarity.

Regularization

In this section is we apply the regularization procedure introduced in [43] solution u of (1.2.1). In this way we obtain an equation satisfied in the pointwise a.e. sense, all of whose terms are integrable. Although the technique is applicable in greater generality, in practice we will only apply the regularization to solutions of (1.2.1) when $\Omega = B_1(0)$ (see Remarks 2.3.9 and 2.3.10). For the sake of clarity, especially regarding notation, we recall some of the relevant definitions and properties, noting that we do not seek full generality in what follows.

Definition 2.3.4. *For $u \in C(\overline{\Omega} \times [0,T])$ and $\epsilon, \kappa > 0$, define*

$$u_{\epsilon,\kappa}(x,t) = \inf_{(y,s)\in\Omega\times(0,T)} \left(u(y,s) + \frac{1}{2\epsilon}|x-y|^2 + \frac{1}{2\kappa}|t-s|^2 \right), \tag{2.3.3}$$

$$u^{\epsilon}(x,t) = \sup_{y\in\Omega} \left(u(y,t) - \frac{1}{2\epsilon}|x-y|^2 \right). \tag{2.3.4}$$

We may also define $u^{\epsilon,\kappa}$ and u_ϵ similarly. Note that we use just one index when the convolution is performed in the space variable only. In the following statement we collect a series of well-known facts regarding these operations which will be used shortly hereafter.

Proposition 2.3.5. *Assume $u \in C(\overline{\Omega} \times [0,T])$, and let $\epsilon, \kappa, \delta > 0$.*

(i) Both operations preserve both pointwise upper and lower bounds, i.e.,

$$\inf u \le u_{\epsilon,\kappa} \le \sup u,$$
$$\inf u \le u^{\epsilon} \le \sup u,$$

where inf *and* sup *are taken over $\Omega \times (0,T)$.*

(ii) Let $\epsilon^* = 2\sqrt{\epsilon\|u\|_\infty}$, $\kappa^* = 2\sqrt{\kappa\|u\|_\infty}$, $\Omega^{\epsilon^*} = \{x \in \Omega \mid d(x,\partial\Omega) > \epsilon^*\}$. For all $(x,t) \in \Omega^{\epsilon^*} \times (\kappa^*, T-\kappa^*)$, there exist $(y,s) \in \Omega \times (0,T)$ such that

$$u_{\epsilon,\kappa}(x,t) = u(y,s) + \frac{1}{2\epsilon}|x-y|^2 + \frac{1}{2\kappa}|t-s|^2.$$

In other words, the sup *and* inf *in the definition of the convolutions are achieved, provided we are at a sufficient distance from the boundary.*

(iii) *Both* $u_{\epsilon,\kappa}$ *and* $u^{\epsilon,\kappa}$ *are Lipschitz continuous in* x *with constant* $\frac{K}{\sqrt{\epsilon}}$*, where* $K = 2\|u\|_\infty$*. That is,*

$$\sup_{\substack{x,y\in\Omega \\ t\in[0,T]}} \frac{|u(x,t)-u(y,t)|}{|x-y|} \le \frac{K}{\sqrt{\epsilon}}.$$

Similarly, they are Lipschitz continuous in t *with constant* $\frac{K}{\sqrt{\kappa}}$*.*

(iv) $u^{\epsilon,\kappa}, u_{\epsilon,\kappa} \to u$ *uniformly as* $\epsilon, \kappa \to 0$*, and similarly for* u^ϵ*.*

(v) $u^{\epsilon,\kappa}, u_{\epsilon,\kappa}$ *are respectively semiconvex and semiconcave. In particular, they are twice differentiable a.e. That is, there are measurable functions* $a : \Omega \times [0,T] \to \mathbb{R}$*,* $q : \Omega \times [0,T] \to \mathbb{R}^N$*,* $M : \Omega \times [0,T] \to S(N)$ *such that*

$$\begin{aligned} u^{\epsilon,\kappa}(y,s) &= u^{\epsilon,\kappa}(x,t) + a(x,t)(s-t) + \langle q(x,t), y-x\rangle \\ &\quad + \langle M(x,t)(y-x), y-x\rangle + o(|y-x|^2 + |s-t|). \end{aligned}$$

We will denote $a = (u^{\epsilon,\kappa})_t$*,* $q = Du^{\epsilon,\kappa}$*,* $M = D^2u^{\epsilon,\kappa}$ *for simplicity. The same goes for* $u_{\epsilon,\kappa}$*.*

(vi) *With the notation above,*

$$D^2 u_{\epsilon,\kappa} \le \frac{1}{\epsilon} I \quad \text{and} \quad D^2 u^{\epsilon,\kappa} \ge -\frac{1}{\epsilon} I \quad \text{a.e. in } \Omega \times [0,T]. \tag{2.3.5}$$

(vii) $(u_{\epsilon,\kappa})_\delta = u_{\epsilon+\delta,\kappa}$*.*

(viii) $(u_{\epsilon+\delta,\kappa})^\delta \le u_{\epsilon,\kappa}$*.*

(ix) *The operation* $u \mapsto (u_\delta)^\delta$ *preserves semiconcavity, i.e., if* u *is* $\frac{1}{2\epsilon}$*-semiconcave, then* $(u_\delta)^\delta$ *is* $\frac{1}{2\epsilon}$*-semiconcave.*

Remark 2.3.6. The easier proofs follow more or less directly from the definitions (see e.g., [30]), while (vii) and (viii) may be found in [24]. Property (v) uses the well-known theorems of Rademacher and Alexandrov on the differentiability of Lipschitz and convex functions, respectively; see [32] and the Appendix of [25].

The time-independent version of the following result appears as Lemma 4.2 in [55]. We say that F is proper if for all $(X,\xi) \in S(N) \times \mathbb{R}^N$, $r, s \in \mathbb{R}$, if $r \le s$ then $F(X,\xi,r) \le F(X,\xi,s)$.

Lemma 2.3.7. *Let* u *be a viscosity supersolution of* $u_t + F(D^2u, Du, u) = 0$ *in* $\Omega \times (0,T)$*, where* F *is proper. Then, using the notation of Proposition 2.3.5,* $u_{\epsilon,\kappa}$ *is a viscosity supersolution of* $u_t + F(D^2u, Du, u) = 0$ *in* $\Omega^{\epsilon^*} \times (\kappa^*, T-\kappa^*)$*.*

Proof: Let $\varphi = \varphi(x,t)$ be a C^2 function that touches $u_{\epsilon,\kappa}$ from below at $(\hat{x},\hat{t}) \in \Omega^{\epsilon^*} \times (\kappa^*, T-\kappa^*)$, that is, $\varphi(\hat{x},\hat{t}) = u_{\epsilon,\kappa}(\hat{x},\hat{t})$ and for $|x-\hat{x}|+|t-\hat{t}| < \delta$ and sufficiently small $\delta > 0$,

$$\varphi(x,t) \leq u_{\epsilon,\kappa}(x,t). \tag{2.3.6}$$

By Proposition 2.3.5, (ii) there exist $(\hat{y},\hat{s}) \in \Omega \times (0,T)$ such that

$$u_{\epsilon,\kappa}(\hat{x},\hat{t}) = u(\hat{y},\hat{s}) + \frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 + \frac{1}{2\kappa}|\hat{t}-\hat{s}|^2,$$

with $(\hat{y},\hat{s}) \to (\hat{x},\hat{t})$ as $\epsilon, \kappa \to 0$. Hence, for sufficiently small ϵ, κ, $(\hat{y},\hat{s})$ remains close to (x,t) as in (2.3.6). Therefore,

$$\varphi(x,t) \leq u_{\epsilon,\kappa}(x,t) \leq u(x+(\hat{y}-\hat{x}), t+(\hat{s}-\hat{t})) + \frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 + \frac{1}{2\kappa}|\hat{t}-\hat{s}|^2.$$

Evaluating this expression now at $(x+(\hat{x}-\hat{y}), t+(\hat{t}-\hat{s}))$, we have that

$$\tilde{\varphi}(x,t) := \varphi(x+(\hat{x}-\hat{y}), t+(\hat{t}-\hat{s})) - \frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 - \frac{1}{2\kappa}|\hat{t}-\hat{s}|^2 \leq u(x,t).$$

We also have, from the choice of $(\hat{y},\hat{s})$, that $\tilde{\varphi}(\hat{y},\hat{s}) = u(\hat{y},\hat{s})$. Hence $\tilde{\varphi}$ is a valid test function for u. Since

$$\begin{aligned} &D^2\tilde{\varphi}(\hat{y},\hat{s}) = D^2\varphi(\hat{x},\hat{t}), \quad D\tilde{\varphi}(\hat{y},\hat{s}) = D\varphi(\hat{x},\hat{t}), \\ &\tilde{\varphi}_t(\hat{y},\hat{s}) = \varphi_t(\hat{x},\hat{t}), \quad \tilde{\varphi}(\hat{y},\hat{s}) \leq \varphi(\hat{x},\hat{t}), \end{aligned}$$

by the properness of F,

$$\begin{aligned} &\varphi_t(\hat{x},\hat{t}) + F(D^2\varphi(\hat{x},\hat{t}), D\varphi(\hat{x},\hat{t}), \varphi(\hat{x},\hat{t})) \geq \\ &\quad \tilde{\varphi}_t(\hat{y},\hat{s}) + F(D^2\tilde{\varphi}(\hat{y},\hat{s}), D\tilde{\varphi}(\hat{y},\hat{s}), \tilde{\varphi}(\hat{y},\hat{s})) \geq 0. \end{aligned}$$

Hence, $u_{\epsilon,\kappa}$ is a supersolution. That it is a subsolution is proved similarly. □

The following Proposition is an adaptation of Proposition 4.6 from [24], which is developed in a slightly different context.

Proposition 2.3.8. *Let $\Omega' \subset\subset \Omega$, $0 < t_0 < t_1 < T$ and u be a bounded viscosity supersolution of* (1.2.1) *in $\Omega \times (0,T)$. Then, there exist constants $\epsilon, \delta, \kappa > 0$ such that the regularized function $w = (u_{\epsilon+\delta,\kappa})^\delta$ satisfies*

$$w_t - \mathcal{M}^-(D^2 w) \geq |Dw|^p \quad \textit{a.e. in } \Omega' \times (t_0,t_1). \tag{2.3.7}$$

In particular, w is a so-called L^∞-strong supersolution of (2.3.7).

Proof: We apply Lemma 2.3.7 to (1.2.1) to find that, for sufficiently small ϵ and κ, $u_{\epsilon,\kappa}$, depending on $\|u\|_\infty(\Omega \times (0,T))$ (see Proposition 2.3.5, (ii)) is a viscosity supersolution of (1.2.1) in $\Omega' \times (t_0,t_1)$. Observe that the Lemma applies since there is no x dependence. The regularized function w defined above is both semiconvex and semiconcave in x, as well as Lipschitz-continuous in t. Hence it is twice differentiable a.e. in $\Omega' \times (\tau, T-\tau)$, in the sense of having a second order "parabolic" Taylor expansion (as in Proposition 2.3.5, (v)).

Let $(\hat{x}, \hat{t})$ be any such point of differentiability. As in Proposition 2.3.5, (viii), we have that $w \leq u_{\epsilon,\kappa}$. Suppose that $w(\hat{x}, \hat{t}) = u_{\epsilon,\kappa}(\hat{x}, \hat{t})$. For (x,t) in a neighborhood of $(\hat{x}, \hat{t})$, we then have

$$\begin{aligned} u_{\epsilon,\kappa}(x,t) \geq w(x,t) &= w(\hat{x},\hat{t}) + w_t(\hat{x},\hat{t})(t-\hat{t}) + \langle Dw(\hat{x},\hat{t}), x-\hat{x}\rangle \\ &\quad + \langle D^2 w(\hat{x},\hat{t}), x-\hat{x}\rangle + o(|x-\hat{x}|^2 + |t-\hat{t}|) \\ &= u_{\epsilon,\kappa}(\hat{x},\hat{t}) + w_t(\hat{x},\hat{t})(t-\hat{t}) + \langle Dw(\hat{x},\hat{t}), x-\hat{x}\rangle \\ &\quad + \langle D^2 w(\hat{x},\hat{t}), x-\hat{x}\rangle + o(|x-\hat{x}|^2 + |t-\hat{t}|), \end{aligned}$$

which implies that $(w_t(\hat{x},\hat{t}), Dw(\hat{x},\hat{t}), D^2w(\hat{x},\hat{t})) \in \mathcal{P}^{2,-}u_{\epsilon,\kappa}(\hat{x},\hat{t})$, the parabolic subjet at $(\hat{x},\hat{t})$ (see, e.g., [25]). Since $u_{\epsilon,\kappa}$ is a viscosity supersolution, this gives

$$w_t(\hat{x},\hat{t}) - \mathcal{M}^-(D^2 w(\hat{x},\hat{t})) - |Dw(\hat{x},\hat{t})|^p \geq 0.$$

Assume now that $w(\hat{x},\hat{t}) < u_{\epsilon,\kappa}(\hat{x},\hat{t})$. In this case, by Proposition 4.4 in [24], $D^2 w(\hat{x},\hat{t})$ has an eigenvalue equal to $-\frac{1}{\delta}$. On the other hand, by Proposition 4.5 in [24], w is $\frac{1}{2\epsilon}$-semiconvex, so the remaining eigenvalues are bounded by above by $\frac{1}{\epsilon}$. Recalling also the gradient bounds which come from the Lipschitz continuity of w with respect to both x and t, as in Proposition 2.3.5, (iii), we obtain

$$w_t(\hat{x},\hat{t}) - \mathcal{M}^-(D^2 w(\hat{x},\hat{t})) - |Dw(\hat{x},\hat{t})|^p \geq -\frac{K}{\kappa^{\frac{1}{2}}} + \lambda\frac{1}{\delta} - (n-1)\Lambda\frac{1}{\epsilon} - \frac{K^p}{\epsilon^{\frac{p}{2}}}.$$

By taking $\delta = o(\epsilon^{\frac{p}{2}})$ and ϵ sufficiently small, the right-hand side of the above inequality becomes nonnegative. Hence, w is a supersolution. □

Remark 2.3.9. For the proof of Theorem 1.2.2 we apply Proposition 2.3.8 in the case where $\Omega = (0,1)$, and regularization is applied to $U = U(r)$, the radial part of the solution u of (1.2.1) in $B_1(0) \times [0,T]$. The spatial regularization will be performed with respect to the radial variable. To alleviate the notation of Section 2.4, we briefly switch to using $\tilde{\epsilon}$ and $\tilde{\delta}$ for the spatial regularization parameters. Precisely, this gives

$$\begin{aligned} &w(r,t) = (U_{\tilde{\epsilon}+\tilde{\delta},\kappa})^{\tilde{\delta}}(r,t) \\ &= \sup_{r_1\in(0,1)} \inf_{\substack{r_2\in(0,1)\\ s\in(0,T)}} \left(U(r_2,s) + \frac{1}{2(\tilde{\epsilon}+\tilde{\delta})}|r_2-r_1|^2 + \frac{1}{2\kappa}|t-s|^2 - \frac{1}{2\tilde{\delta}}|r-r_1|^2 \right). \end{aligned}$$

Note also that from the proof of Proposition 2.3.8, we choose $\tilde{\delta} = \tilde{\delta}(\tilde{\epsilon})$, with $\tilde{\delta} \to 0$ as $\tilde{\epsilon} \to 0$, so we need only choose suitable $\tilde{\epsilon} > 0$ in the regularization.

Since the viscosity solution of (1.2.1) is uniformly bounded (see Remark 2.1.6), Proposition 2.3.8 provides a supersolution to (1.2.1) on a domain which arbitrarily approaches $(0,1) \times (0,T)$. That is, we can have w satisfy (2.3.7) in $(\epsilon, 1-\epsilon) \times (t_0, t_1)$ for arbitrarily small ϵ and t_0, and t_1 close to T, provided we choose small enough regularization parameters, depending on $\|u_0\|_\infty$. For this reason, in the proof of Theorem 1.2.2 given in the following section, we require certain estimates as $\epsilon,\ t_0 \to 0$. This use of ϵ is maintained from the following subsection onwards, throughout Section 2.4, where additionally δ is used as a different cut-off parameter.

The use of ϵ and δ as regularization parameters is only briefly revisited in Subsection 2.5, where some comments are made regarding the adaptation of Proposition 2.3.8 to an equation with more general nonlinearities.

Remark 2.3.10. We obtain the inequality in divergence form as follows. By combining Proposition 2.3.8, Lemma 2.3.1, and the considerations of Remark 2.3.9, we have w satisfies

$$w_t - \theta(w'')w'' - \frac{N-1}{r}\theta(w')w' - |w'|^p \geq 0 \quad \text{for a.e. } r \in (\epsilon, 1-\epsilon), t \in (t_0, t_1), \tag{2.3.8}$$

for arbitrarily small $\epsilon, t_0 > 0$ and t_1 arbitrarily close to T. Define

$$\hat{N} = \frac{\theta(w')}{\theta(w'')}(N-1) + 1, \quad \rho(r) = e^{\int_{1-\epsilon}^{r} \frac{\hat{N}-1}{s}\, ds}, \quad \text{and} \quad \tilde{\rho}(r) = \frac{\rho(r)}{\theta(w'')}.$$

Note that ρ is the indefinite integral of a measurable function, and is therefore absolutely continuous. In particular, this implies that it is differentiable a.e.

Multiplying (2.3.8) by $\tilde{\rho}$, we obtain, for the second-order terms,

$$\tilde{\rho}\left(\theta(w'')w'' + \frac{N-1}{r}\theta(w')w'\right) = (\rho w')'.$$

Hence,

$$\tilde{\rho} w_t \geq (\rho w')' + \tilde{\rho}|w'|^p \quad \text{for a.e. } r \in (\epsilon, 1-\epsilon), t \in (t_0, t_1). \tag{2.3.9}$$

Remark 2.3.11. The functions ρ and $\tilde{\rho}$ depend on the regularization parameters both explicitly and through the solution of the approximate equation w, but we omit these dependencies for simplicity of notation.

We now provide a couple of bounds which will be useful later. As

$$\hat{N} - 1 = \frac{\theta(w')}{\theta(w'')}(N-1) \leq \frac{\Lambda}{\lambda}(N-1),$$

we have for $\epsilon \in (0, \frac{1}{2})$ and all $r \in (\epsilon, 1-\epsilon)$, that

$$\begin{aligned}\tilde{\rho}(r) &= \frac{1}{\theta(w'')} e^{\int_{1-\epsilon}^{r} \frac{\hat{N}-1}{s}\, ds} \geq \frac{1}{\Lambda}\left(\frac{r}{1-\epsilon}\right)^{\frac{\Lambda}{\lambda}(N-1)} \\ &\geq \frac{1}{\Lambda}\left(\frac{r}{2}\right)^{\frac{\Lambda}{\lambda}(N-1)} := \hat{\rho}(r). \end{aligned}\tag{2.3.10}$$

Note that $\hat{\rho}$ no longer depends on the regularization parameters and is defined for all $r \in (0,1)$. This is the function which appears in the statement of Theorem 1.2.2.

On the other hand, since $\frac{r}{1-\epsilon} \leq 1$ for all $r \in (\epsilon, 1-\epsilon)$, we similarly obtain

$$\tilde{\rho}(r) \leq \frac{1}{\lambda}. \tag{2.3.11}$$

We can also explicitly compute

$$\rho(1-\epsilon) = 1 \quad \text{and} \quad 0 \leq \rho(\epsilon) \leq \left(\frac{\epsilon}{1-\epsilon}\right)^{\frac{\lambda(N-1)}{\Lambda}} \to 0 \quad \text{as } \epsilon \to 0. \tag{2.3.12}$$

Eigenvalue problem for the Pucci extremal operator

The proof of Theorem 1.2.2 involves the solution to the Dirichlet eigenvalue problem for the extremal operator $-\mathcal{M}^-$ in annular domains approximating the punctured ball $B_1(0)\backslash\{0\}$.

More precisely, let $A_\epsilon = B_{1-\epsilon}(0)\backslash\overline{B_\epsilon(0)}$, and consider

$$\begin{cases} -\mathcal{M}^-(D^2\varphi) = \lambda\varphi & \text{in } A_\epsilon, \\ \varphi = 0 & \text{on } \partial A_\epsilon. \end{cases} \tag{2.3.13}$$

Here the boundary condition is satisfied in the classical sense. Note that A_ϵ corresponds to the spatial domain where (2.3.9) is satisfied.

By Proposition 1.1 in [21], there exists a solution pair $(\lambda_1^\epsilon, \varphi_1^\epsilon)$ of (2.3.13) with $\lambda_1^\epsilon > 0$, $\varphi_1^\epsilon \in C^2(A_\epsilon) \cap C(\overline{A_\epsilon})$ and $\varphi_1^\epsilon > 0$ in A_ϵ, where φ_1^ϵ is unique up to a positive constant. We normalize this solution so that $\varphi_1^\epsilon(\frac{1}{2}) = 1$, for reasons that will become apparent later. Our notation indicates that both λ_1^ϵ and φ_1^ϵ depend on the parameters of the spatial regularization through the domain A_ϵ.

We employ the following lemma to state our main theorem without reference to these regularization parameters.

Lemma 2.3.12. *Let $K \subset (0,1)$ be a closed interval such that $[1/4, 3/4] \subset K$. There exists a function $\hat{\varphi} \in C(K)$, such that $\hat{\varphi}(r) > 0$ for all $r \in K$ and $\varphi_1^\epsilon \to \hat{\varphi}$ uniformly over K, up to a subsequence.*

Proof. In general, if we denote by $\lambda_1(\Omega)$ the corresponding *principal half-eigenvalue* (i.e., solution of (2.3.13)) in Ω, we have that $\lambda_1(\Omega') \leq \lambda_1(\Omega)$ if $\Omega \subset \Omega'$; see Proposition 1.1 (*iii*) in [21]. We therefore have the monotonicity $\lambda_1^{\epsilon'} \leq \lambda_1^\epsilon$ if $\epsilon' \leq \epsilon$. For the same reason, $\lambda_1(B_1(0)) \leq \lambda_1^\epsilon$ for all $\epsilon > 0$. Hence, $\lambda_1^\epsilon \to \hat{\lambda}$ as $\epsilon \to 0$, for some $\hat{\lambda} > 0$. Note also that $\lambda_1^\epsilon \leq \lambda((1/4, 3/4))$ for all $\epsilon < 1/4$.

Consider now a closed interval $K' \supset K$. By Harnack's inequality (see Theorem 3.6 in [50]), for all $\epsilon > 0$ small enough such that $K' \subset (\epsilon, 1-\epsilon)$, we have

$$\sup_{K'} \varphi_1^\epsilon \leq \sup_{(\epsilon,1-\epsilon)} \varphi_1^\epsilon \leq C \inf_{(\epsilon,1-\epsilon)} \varphi_1^\epsilon \leq C \inf_{K'} \varphi_1^\epsilon \leq C, \tag{2.3.14}$$

where we used $\varphi_1^\epsilon(\frac{1}{2}) = 1$ and $\frac{1}{2} \in K'$ for the last inequality. The Harnack constant C above depends only on $N, \lambda, \Lambda, \lambda_1^\epsilon$ and $\text{dist}(K', \partial(0,1))$. Since λ_1^ϵ is uniformly bounded for $\epsilon < 1/4$, C is independent of ϵ as well.

It follows that the functions φ_1^ϵ are uniformly bounded, and therefore satisfy a family of ODEs with uniformly bounded right-hand sides. More precisely, using (2.2.2) once more, we have

$$\mathcal{M}^-(D^2\varphi_1^\epsilon) = \theta[(\varphi_1^\epsilon)''](\varphi_1^\epsilon)'' + \theta[(\varphi_1^\epsilon)'](\varphi_1^\epsilon)'\frac{N-1}{r} = -\lambda_1^\epsilon\varphi_1^\epsilon \quad \text{in int}(K'), \tag{2.3.15}$$

with $\|\lambda_1^\epsilon\varphi_1^\epsilon\|_\infty \le C\max\{\hat\lambda, \lambda((1/4, 1/4))\} := \overline{C}$, where C is the Harnack constant above. We proceed with a compactness argument, following [31].

Let $\gamma_\epsilon := \|\varphi_1^\epsilon\|_{C^1(K')}$ and define $\tilde{\varphi^\epsilon} = \varphi_1^\epsilon/\gamma_\epsilon$. Using that $\mathcal{M}^-$ is positive homogeneous, we have that $\tilde{\varphi^\epsilon}$ is also a solution of (2.3.15), and since $\|\tilde{\varphi^\epsilon}\|_{C^1(K')} = 1$ for all $\epsilon < 1/4$, this implies $(\tilde{\varphi^\epsilon})''$ is uniformly bounded as well.

By compactness, this implies that, up to a subsequence, $\tilde{\varphi^\epsilon} \to \tilde\varphi$ uniformly on K' for some $\tilde\varphi \in C^2(\text{int}(K')) \cap C^1(K')$. Recalling that $\lambda_1^\epsilon \to \hat\lambda$, we pass to the limit to find

$$\mathcal{M}^-(D^2\tilde\varphi) = -\hat\lambda\tilde\varphi \quad \text{in int}(K'). \tag{2.3.16}$$

Note $\tilde\varphi \not\equiv 0$ since it is the limit of $\tilde{\varphi^\epsilon}$ and $\|\tilde{\varphi^\epsilon}\|_{C^1(K')} = 1$ for all $\epsilon < 1/4$.

Assume that γ_ϵ becomes unbounded as $\epsilon \to 0$. As before, using the homogeneity of $\mathcal{M}^-$, we see that

$$\mathcal{M}^-(D^2\tilde{\varphi^\epsilon}) = -\frac{\lambda_1^\epsilon\varphi_1^\epsilon}{\gamma_\epsilon} \quad \text{in int}(K'),$$

and $\|\frac{\lambda_1^\epsilon\varphi_1^\epsilon}{\gamma_\epsilon}\|_\infty \le \overline{C}/\gamma_\epsilon \to 0$ as $\epsilon \to 0$. Passing to the limit, this gives $\mathcal{M}^-(D^2\tilde\varphi) = 0$, in contradiction with (2.3.16). It follows that $\gamma_\epsilon = \|\varphi_1^\epsilon\|_{C^1(K')}$ is uniformly bounded in ϵ, hence we can pass to the limit as before (*without* the normalization $\tilde{\varphi^\epsilon}$), to find that $\varphi_1^\epsilon \to \hat\varphi$ uniformly in K' for some $\hat\varphi \in C(K')$ which is a solution of (2.3.16).

From the uniform convergence and $\varphi_1^\epsilon \ge 0$ for all $\epsilon > 0$, we conclude $\hat\varphi \ge 0$, hence the strong maximal principle applies (see Lemma 3.4 in [36]). Recalling also that $\varphi_1^\epsilon(\frac{1}{2}) = 1$, we have $\hat\varphi \not\equiv 0$. Combining these facts, we conclude that $\hat\varphi > 0$ in $\text{int}(K') \supset K$. □

Remark 2.3.13. Since K is a closed interval, $\hat\varphi$ is bounded by below on K by a positive constant which does not depend on ϵ.

The proof of Theorem 1.2.2 requires two additional lemmas.

Lemma 2.3.14. *Let φ_1^ϵ be the solution of* (2.3.13)*, as defined above. Then, for any $0 < \alpha < 1$, there exists a positive constant $C > 0$ such that*

$$\int_\epsilon^{1-\epsilon} (\varphi_1^\epsilon)^{-\alpha}\tilde\rho\, dr < C. \tag{2.3.17}$$

Furthermore, C may be taken uniformly for $\epsilon \in (0, \frac{1}{4})$.

Proof. By Remark 2.3.13, it is possible to bound φ_1^ϵ by below by a positive constant uniformly for small ϵ over a closed interval $K \subset (0,1)$, to be chosen later. Hence, to obtain (2.3.17) it is sufficient to bound the integral near the endpoints ϵ and $1-\epsilon$.

We now proceed as in the proof of Hopf's lemma to obtain a uniform lower bound for $(\varphi_1^\epsilon)'(\epsilon)$. Let $\beta > 0$ and

$$v(r) = \frac{e^{-\beta(\frac{1}{2}-r)^2} - e^{-\beta(\frac{1}{2}-\epsilon)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \qquad \text{for} \qquad \epsilon < r < \frac{1}{2}.$$

We verify that $v \geq 0, v(\epsilon) = 0, v(\frac{1}{2}) = 1$, and compute

$$v'(r) = \frac{2\beta(\frac{1}{2}-r)\, e^{-\beta(\frac{1}{2}-r)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \geq 0, \tag{2.3.18}$$

$$v''(r) = \frac{\left(4\beta^2(\frac{1}{2}-r)^2 - 2\beta\right) e^{-\beta(\frac{1}{2}-r)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \geq 0, \tag{2.3.19}$$

where the inequality in (2.3.19) follows from taking a sufficiently large $\beta > 0$.

We abuse notation slightly and define $v(x) = v(|x|)$ in $B_{\frac{1}{2}}(0) \backslash \overline{B_\epsilon(0)}$. By the previous computation, using also that $\lambda_1^\epsilon > 0, v \geq 0$, we have

$$\mathcal{M}^-(D^2 v) = \lambda v'' + \lambda \frac{n-1}{r} v' \geq 0 \geq -\lambda_1^\epsilon v.$$

Hence v is a subsolution of (2.3.13). Since $v(\epsilon) = \varphi_1^\epsilon(\epsilon), v(\frac{1}{2}) = \varphi_1^\epsilon(\frac{1}{2})$, by comparison we have $v(r) \leq \varphi_1^\epsilon(r)$ for all $\epsilon < r < \frac{1}{2}$ (see, for example, Appendix A in [3]). Recalling (2.3.18), for all $0 < \epsilon < \frac{1}{4}$ this gives

$$(\varphi_1^\epsilon)'(\epsilon) \geq v'(\epsilon) = \frac{2\beta(\frac{1}{2}-\epsilon)\, e^{-\beta(\frac{1}{2}-\epsilon)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \geq \frac{\beta e^{-\frac{\beta}{4}}}{1 - e^{-\frac{\beta}{4}}} =: C. \tag{2.3.20}$$

Note that the last constant does not depend on ϵ.

By looking at the first order expansion of φ_1^ϵ at ϵ,

$$\varphi_1^\epsilon(r) = (\varphi_1^\epsilon)'(\epsilon)(r-\epsilon) + o(|r-\epsilon|),$$

we have that there exists a $\delta > 0$ such that

$$\varphi_1^\epsilon(r) > \frac{(\varphi_1^\epsilon)'(\epsilon)}{2}(r-\epsilon) \quad \text{for all} \quad \epsilon < r < \epsilon + \delta. \tag{2.3.21}$$

In the above series expansion, the constant $\delta > 0$ depends only $(\varphi_1^\epsilon)'(\epsilon)$. In view of (2.3.20), we need only bound $(\varphi_1^\epsilon)'(\epsilon)$ by above, independently of ϵ. For this we use a barrier type argument, taking advantage of some of the computations from Sec. 2.2. Define

$$\psi(r) = A\left(1 - e^{-\gamma(r-\epsilon)}\right),$$

where $A, \gamma > 0$ are to be chosen. We have $\psi(\epsilon) = 0$, and for an appropriate choice of A,

$$\psi(1/2) = A\left(1 - e^{-\gamma(1/2-\epsilon)}\right) \geq A(1 - e^{-\gamma/4}) > 1,$$

again using $\epsilon < 1/4$. Computing as in (2.2.12), and using once more that $\lambda_1^\epsilon \leq \lambda((1/4, 3/4))$, we can choose γ large enough so that

$$\begin{aligned}
&-\mathcal{M}^-(D^2\psi) - \lambda_1^\epsilon\psi \geq -\mathcal{M}^-(D^2\psi) - \lambda((1/4, 3/4))\psi \\
&\quad = A\left(\Lambda\gamma^2 - \lambda\gamma\frac{n-1}{r+\epsilon} + \lambda((1/4, 3/4))\right)e^{-\gamma(r-\epsilon)} - A\lambda((1/4, 3/4)) \\
&\quad \geq A\left(\Lambda\gamma^2 - \lambda\gamma\frac{n-1}{r}\right)e^{-\gamma r} - A\lambda((1/4, 3/4)) \geq 0.
\end{aligned}$$

Thus, by comparison, $\psi \geq \varphi_1^\epsilon$ in $[\epsilon, 1-\epsilon]$, for all $\epsilon < 1/4$. Hence,

$$(\varphi_1^\epsilon)'(\epsilon) \leq \psi'(\epsilon) = A\gamma, \tag{2.3.22}$$

and from this we conclude that δ in (2.3.21) does not depend on ϵ.

We then estimate, for $\epsilon < r < \epsilon + \delta$,

$$\begin{aligned}
(\varphi_1^\epsilon(r))^{-\alpha} &< \left(\frac{(\varphi_1^\epsilon)'(\epsilon)}{2}\right)^{-\alpha}(r-\epsilon)^{-\alpha} \\
&\leq (C/2)^{-\alpha}(r-\epsilon)^{-\alpha},
\end{aligned}$$

where we have used (2.3.20) for the second inequality. Recall the bound $\tilde{\rho}(r) \leq \frac{1}{\lambda}$ given by (2.3.11). Then,

$$\begin{aligned}
\int_\epsilon^{\epsilon+\delta} (\varphi_1^\epsilon(r))^{-\alpha}\tilde{\rho}\,dr &\leq (C/2)^{-\alpha}\|\tilde{\rho}\|_\infty \int_\epsilon^{\epsilon+\delta}(r-\epsilon)^{-\alpha}\,dr \\
&\leq (C/2)^{-\alpha}\,\frac{1}{\lambda}\int_0^\delta r^{-\alpha}\,dr < \tilde{C}.
\end{aligned}$$

A similar bound can be obtained over $1-\epsilon-\delta < r < 1-\epsilon$. We may then choose $K = [\delta, 1-\delta]$ with δ as above and recall that φ_1^ϵ are uniformly bounded by below on K. Also note that for all $\epsilon > 0$,

$$K = [\delta, 1-\delta] \supset (\epsilon+\delta, 1-\epsilon-\delta), \tag{2.3.23}$$

hence, we may combine the bounds near the endpoints with the lower bound on K to obtain (2.3.17). □

Remark 2.3.15. The interval $K = [\delta, 1-\delta]$ is the one that appears in the statement of Theorem 1.2.2. As claimed, δ depends only on λ, Λ, n.

Lemma 2.3.16. *Consider $\rho : [\epsilon, 1-\epsilon] \to \mathbb{R}$ as defined in Remark 2.3.10. Let $v : [\epsilon, 1-\epsilon] \to \mathbb{R}$ be once differentiable such that $v(1-\epsilon) = 0$. Then,*

$$\int_\epsilon^{1-\epsilon} |v|\,\rho\,dx \leq \int_\epsilon^{1-\epsilon} |v'|\,\rho\,dx. \tag{2.3.24}$$

In the proof of the above inequality we employ the following result from [59]:

Theorem 2.3.17. *Let $\nu : [0,1] \to \mathbb{R}$ be a non-negative, non-vanishing, continuous weight on the closed unit interval. Let $f : [0,1] \to \mathbb{R}$ be once differentiable and satisfy $f(0) = 0$. Then,*

$$\int_0^1 |f(x)|\nu(x)\,dx \le \left(\max_{0\le x\le 1} \frac{1}{\nu(x)} \int_x^1 \nu(z)\,dz\right) \int_0^1 |f'(x)|\nu(x)\,dx, \qquad (2.3.25)$$

and the constant is sharp.

Proof of Lemma 2.3.16: Let v be as in the statement of the Lemma, and define $g : [0,1] \to \mathbb{R},\ g(r) = (1-2\epsilon)r + \epsilon$. Note that g is an affine change of variables sending $[0,1]$ to $[\epsilon, 1-\epsilon]$ and that $g'(r) = 1-2\epsilon$. Define

$$f(r) = v(g(1-r)), \qquad \nu(r) = \rho(g(1-r)).$$

We have that $f(0) = v(g(1)) = v(1-\epsilon) = 0$, hence f so defined satisfies the hypotheses of Theorem 2.3.17. The weight ρ is continuous and non-negative on $[0,1]$, and furthermore, from the computations in Remark 2.3.11, for all $r \in [\epsilon, 1-\epsilon]$

$$\rho(r) \ge \left(\frac{r}{2}\right)^{\frac{\Lambda}{\lambda}(N-1)} > 0,$$

i.e., ρ is also non-vanishing.

By changing variables, we obtain

$$\begin{aligned}\int_0^1 |f(r)|\nu(r)\,dx &= \int_0^1 |v(g(1-r))|\rho(g(1-r))\,dr \\ &= \frac{1}{1-2\epsilon}\int_\epsilon^{1-\epsilon} |v(r)|\rho(r)\,dr,\end{aligned}$$

and similarly,

$$\begin{aligned}\int_0^1 |f'(r)|\nu(r)\,dr &= \int_0^1 |v'(g(r))||g'(r)|\rho(g(1-r))\,dr \\ &= \int_\epsilon^{1-\epsilon} |v'(r)|\rho(r)\,dr.\end{aligned}$$

Next, we estimate the constant in (2.3.25). It is easy to check that ρ is strictly increasing from the definition, as is g. Hence, for $s \ge r$ we have $\rho(g(1-s)) \le \rho(g(1-r))$. Therefore, for all $0 \le r \le 1$,

$$\begin{aligned}\frac{1}{\nu(r)}\int_r^1 \nu(s)\,ds &= \frac{1}{\rho(g(1-r))}\int_r^1 \rho(g(1-s))\,ds \\ &\le \frac{1}{\rho(g(1-r))}\rho(g(1-r))(1-r) \\ &\le 1-r \le 1.\end{aligned}$$

By the above computations, we can apply (2.3.25) to obtain

$$\int_{\epsilon}^{1-\epsilon} |v(r)|\,\rho(r)\,dx \le \int_{\epsilon}^{1-\epsilon} |v'(r)|\,\rho(r)\,dr.$$

□

Remark 2.3.18. In the proof of Theorem 1.2.2 we will actually use Lemma 2.3.16 with the weight $\tilde{\rho}$ instead of ρ. This is possible since

$$\frac{1}{\Lambda}\rho \le \tilde{\rho} \le \frac{1}{\lambda}\rho.$$

The preceding argument, however, does not apply to $\tilde{\rho}$ directly since it is not continuous.

2.4 Nonexistence of global solutions and LOBC

We now prove our main result in the radial case, i.e., when the spatial domain is a ball and the initial data is radially symmetric. The proof of the result in a general domain follows more or less easily from the radial case.

Proof of Theorem 1.2.2: Our proof uses key ideas from that of Theorem 2.1 in [57]. Some care is required in choosing the constants appearing in our argument in the correct order. Specifically, we first choose u_0 large in an appropriate sense, then choose the regularization parameters sufficiently small. This amounts to making ϵ and t_0 approach 0, although the actual limit is not taken. This difficulty is not present in [57], since the solutions dealt with therein are classical and no regularization is needed.

Consider the differential inequality

$$\dot{y}(t) \ge Cy(t)^p, \quad 0 < t_0 < t < t_1, \tag{2.4.1}$$

$$y(t_0) = M_0, \tag{2.4.2}$$

where C, $M_0 > 0$. We can integrate (3.4.1) explicitly to obtain

$$0 \le y(t)^{1-p} \le C(1-p)(t-t_0) + M_0^{1-p}.$$

Hence, $y(t)^{1-p} \to 0$ as $t \to t_0 + \frac{M_0^{1-p}}{C(p-1)}$. Since $1-p < 0$, this implies $y(t) \to +\infty$. Alternatively, for a fixed $t_1 > t_0$, blow-up occurs for $t < t_1$ provided we have

$$M_0 > \left[C(p-1)(t_1-t_0)\right]^{-\frac{1}{p-1}}. \tag{2.4.3}$$

So fix $T > 0$ and assume that the viscosity solution u of (1.2.1) in $B_1(0) \times [0,T]$ with radial initial data $u_0 \in C^1(\overline{B_1(0)})$ satisfies (1.1.2) in the classical sense. Writing $u = u(r,t)$, with $r \in [0,1]$ and $t \in [0,T]$, this means $u(1,t) = 0$ for all $t \in [0,T]$. We will specify the largeness condition on u_0 in terms of M_0

later, but may consider it set from now on, since it depends only on constants already available.

Recall now the regularized function w defined in Remark 2.3.9, which satisfies the inequality (2.3.9) in $(\epsilon, 1-\epsilon) \times (t_0, t_1)$. We take ϵ so that $0 < \epsilon < \delta$, where δ is the same constant given above. This is so that $[\delta, 1-\delta] \subset (\epsilon, 1-\epsilon)$. Note also that the regularization in time may be performed so that t_0 and t_1 are arbitrarily close to 0 and T, respectively (see Remarks 2.3.9 and 2.3.10). It follows that M_0 depends only on p, T, and the coefficient C in (3.4.1).

Using the solution pair (λ_1, φ_1) to the eigenvalue problem (2.3.13), we define

$$z(t) = \int_\epsilon^{1-\epsilon} w(r,t)\varphi_1(r)\tilde{\rho}(r)\,dr, \quad t \in (t_0, t_1).$$

We will to show that $z = z(t)$ satisfies (3.4.1) with $z(t_0) \geq M_0$, and consequently blows up for some $t < t_1$. This is a contradiction, since z is uniformly bounded for all $t \geq t_0$ by the uniform convergence of $w \to u$ and the fact that u , φ_1, and $\tilde{\rho}$ are all uniformly bounded (see Remark 2.1.6, (2.3.14) and (2.3.11), respectively). Therefore, the solution u cannot satisfy the boundary data in the classical sense for all time. In other words, LOBC occurs.

Using (2.3.9), we compute

$$\begin{aligned} \dot{z}(t) &= \int_\epsilon^{1-\epsilon} w_t(r,t)\varphi_1(r)\tilde{\rho}\,dr \geq \int_\epsilon^{1-\epsilon} \left((\rho w')' + \tilde{\rho}|w'|^p\right)\varphi_1(r)\,dr \\ &= \int_\epsilon^{1-\epsilon} (\rho w')'\varphi_1(r)\,dr + \int_\epsilon^{1-\epsilon} |w'|^p\varphi_1(r)\tilde{\rho}\,dr \\ &=: I_1 + I_2. \end{aligned} \tag{2.4.4}$$

(We omit some of the functions' arguments for simplicity.) Integrating by parts twice in I_1, we obtain

$$I_1 = \int_\epsilon^{1-\epsilon} w(r,t)\,(\rho\varphi_1')'\,dr + \rho w'\varphi_1|_\epsilon^{1-\epsilon} - \rho w\varphi_1'|_\epsilon^{1-\epsilon}.$$

Since $\varphi_1(\epsilon) = \varphi_1(1-\epsilon) = 0$, we have that $\rho w'\varphi_1|_\epsilon^{1-\epsilon} = 0$. On the other hand, $\varphi_1'(\epsilon) > 0$, $\varphi_1'(1-\epsilon) < 0$ and $\rho, w \geq 0$ imply that $-\rho w\varphi_1'|_\epsilon^{1-\epsilon} \geq 0$. Hence, we continue estimating

$$\begin{aligned} I_1 &\geq \int_\epsilon^{1-\epsilon} w(r,t)\,(\rho\varphi_1')'\,dr \geq \int_\epsilon^{1-\epsilon} w(r,t)\,(\tilde{\rho}\mathcal{M}^-(D^2\varphi_1))\,dr \\ &= \int_\epsilon^{1-\epsilon} w(r,t)\,(-\lambda_1\varphi_1)\tilde{\rho}\,dr = -\lambda_1 z(t). \end{aligned} \tag{2.4.5}$$

The second inequality above comes from the minimality of the Pucci operator. Indeed, for all radial $\varphi \in C^2$, by the definition of the weights ρ and $\tilde{\rho}$, we have

$$\frac{1}{\tilde{\rho}}(\rho\,\varphi')' = \theta(w'')\varphi'' + \theta(w')\varphi'\frac{n-1}{r}$$

wherever w'' is defined. This defines an elliptic operator with ellipticity constants λ, Λ. Therefore, $\mathcal{M}^-(D^2\varphi) \leq {}^1\!/_{\tilde\rho}(\rho\,\varphi')'$ a.e. in $(\epsilon, 1-\epsilon)$.

We turn to estimating I_2. From Hölder's inequality for the measure $\tilde\rho(r)\,dr$,

$$\int_\epsilon^{1-\epsilon} |w'|\,\tilde\rho\,dr = \int_\epsilon^{1-\epsilon} |w'|(\varphi_1)^{\frac{1}{p}}(\varphi_1)^{-\frac{1}{p}}\,\tilde\rho\,dr$$
$$\leq \left(\int_\epsilon^{1-\epsilon} (\varphi_1)^{-\frac{1}{p-1}}\,\tilde\rho\,dr\right)^{\frac{p-1}{p}} \left(\int_\epsilon^{1-\epsilon} |w'|^p\varphi_1\tilde\rho\,dr\right)^{\frac{1}{p}}. \quad (2.4.6)$$

The assumption $p > 2$ implies ${}^1\!/_{p-1} \in (0,1)$, so we can apply Lemma 2.3.14 to bound the first integral in the right-hand side of (2.4.6) by a constant C that does not depend on ϵ. We then have

$$\int_\epsilon^{1-\epsilon} |w'|\tilde\rho\,dr \leq C\left(\int_\epsilon^{1-\epsilon} |w'|^p\varphi_1\tilde\rho\,dr\right)^{\frac{1}{p}} = CI_2^{\frac{1}{p}}. \quad (2.4.7)$$

Define $\tilde w(r,t) := w(r,t) - w(1-\epsilon)$ for all $r \in [\epsilon, 1-\epsilon],\ t \in (t_0, t_1)$. Note that $\tilde w' = w'$, and $\tilde w(1-\epsilon, t) = 0$, hence Poincaré's inequality (Lemma 2.3.16) applies. Moreover, since we are taking $u_0 \geq 0$, $u_0 \not\equiv 0$, the strong minimum principle for u the viscosity solution of (1.2.1) in $B_1(0) \times [0,T]$ implies that $u(0,t) > 0$ for all $t > 0$ (see e.g., [26]). Therefore, by the uniform convergence of $w \to u$, we have that

$$w(\epsilon, t) \to u(0,t) > 0, \quad \text{for all } t > t_0, \text{ as } \epsilon \to 0.$$

On the other hand, the uniform convergence $w \to u$ and the uniform continuity of u in $\overline{B_1(0)} \times [0,T]$ imply that $w(1-\epsilon) \to 0$ as $\epsilon \to 0$. Indeed, let $\nu > 0$. By assumption $u(1,t) = 0$ for all $t \geq t_0$. Hence, for small $\epsilon > 0$,

$$w(1-\epsilon,t) = w(1-\epsilon,t) - u(1,t) \leq |w(1-\epsilon,t) - u(1-\epsilon,t)|$$
$$+ |u(1-\epsilon,t) - u(1,t)| < 2\nu. \quad (2.4.8)$$

(We will later simply write $w(1-\epsilon) = o(1)$.) Again by the minimum principle (w is an L^∞-strong solution, as shown in Proposition 2.3.8, hence also a viscosity solution; see e.g., [24]),

$$\min_{[\epsilon,1-\epsilon]} w(\cdot,t) = \min\{w(\epsilon,t), w(1-\epsilon,t)\}.$$

Together with the considerations above, this implies that $\min_{[\epsilon,1-\epsilon]} w(\cdot,t) = w(1-\epsilon,t)$. Therefore, $\tilde w \geq 0$.

Thus, applying Lemma 2.3.16,

$$\int_\epsilon^{1-\epsilon} (w(r,t) - w(1-\epsilon,t))\tilde\rho\,dr = \int_\epsilon^{1-\epsilon} \tilde w(r,t)\,\tilde\rho\,dr \leq \int_\epsilon^{1-\epsilon} |\tilde w'|\,\tilde\rho\,dr = \int_\epsilon^{1-\epsilon} |w'|\,\tilde\rho\,dr.$$

Since $\tilde\rho$ is uniformly bounded (see (2.3.11)), this gives

$$\int_\epsilon^{1-\epsilon} w(r,t)\,\tilde\rho\,dr \leq Cw(1-\epsilon,t) + \int_\epsilon^{1-\epsilon} |w'|\,\tilde\rho\,dr.$$

Recalling (2.3.14) and using the elementary inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$, we have

$$\begin{aligned} z(t)^p &= \left(\int_\epsilon^{1-\epsilon} w(r,t)\,\varphi_1(r)\,\tilde\rho\,dr\right)^p \le C\left(\int_\epsilon^{1-\epsilon} w(r,t)\,\tilde\rho\,dr\right)^p \\ &\le C\left(Cw(1-\epsilon,t)+\int_\epsilon^{1-\epsilon}|w'|\,\tilde\rho\,dr\right)^p \\ &\le C\left[w(1-\epsilon,t)^p+\left(\int_\epsilon^{1-\epsilon}|w'|\,\tilde\rho\,dr\right)^p\right]. \end{aligned}$$

Together with (2.4.7), this implies

$$I_2 \ge C\left(\int_\epsilon^{1-\epsilon}|w'|^p\varphi_1\tilde\rho\,dr\right)^p \ge Cz(t)^p - w(1-\epsilon,t)^p. \tag{2.4.9}$$

Thus, combining (2.4.4), (2.4.5), (2.4.8) and (2.4.9), we have obtained

$$\dot z(t) \ge -\lambda_1 z(t) + Cz(t)^p + o(1), \quad t\in(t_0,\,t_1), \tag{2.4.10}$$

where, crucially, the coefficient C does not depend on either ϵ or u_0.

We can reduce (2.4.10) to (3.4.1) as follows. Using that $\varphi_1 = \varphi_1^\epsilon \to \hat\varphi$ in $[\delta,1-\delta]$ uniformly as $\epsilon\to 0$ (see Lemma 2.3.12), $w\to u$ in $[0,1]\times[0,T]$ uniformly as $\epsilon,\ t_0\to 0$, and the uniform continuity of u (more precisely that $u(\cdot,t_0)\to u_0$ as $t_0\to 0$), the bound $\tilde\rho\ge\hat\rho$ given in (2.3.10), and the fact that all these functions are nonnegative, we have

$$\begin{aligned} z(t_0) &= \int_\epsilon^{1-\epsilon} w(r,t_0)\,\varphi_1(r)\,\tilde\rho(r)\,dr \ge \int_\delta^{1-\delta} w(r,t_0)\,\varphi_1(r)\hat\rho(r)\,dr \\ &\ge \int_\delta^{1-\delta} w(r,t_0)\,\hat\varphi(r)\hat\rho(r)\,dr + o(1) \\ &\ge \int_\delta^{1-\delta} u_0(r)\hat\varphi(r)\hat\rho(r)\,dr + o(1), \end{aligned} \tag{2.4.11}$$

where we have also used that $\hat\varphi$ and $\hat\rho$ are uniformly bounded. Since these functions are also bounded by below in $[\delta,1-\delta]$ by a positive constant, choosing the value of the last integral in (2.4.11) is equivalent to the condition (1.2.2) from the statement of the Theorem.

We recall from the proof of Lemma 2.3.12 that $\lambda_1=\lambda_1^\epsilon$ is also uniformly bounded. Since $p>2$, this implies that the term $Cz(t)^p$ dominates the linear term in (2.4.10). More precisely, if, say $\lambda_1^\epsilon\le C'$, setting

$$\int_\delta^{1-\delta} u_0(r)\hat\varphi(r)\hat\rho(r)\,dr \ge \max\left\{M_0,\left(\frac{2C'}{C}\right)^{\frac{1}{p-1}}\right\}+1,$$

gives both $\dot z \ge (C/2)z(t)^p$ and $z(t_0)\ge M_0$, which is equivalent to (3.4.1). This gives the desired contradiction. □

Remark 2.4.1. The hypothesis which leads to contradiction, i.e, that the solution u satisfies the boundary data in the classical sense, is used only to determine $w(1-\epsilon,t) = o(1)$ in (2.4.8), and in the subsequent application of Lemma (2.3.16). This is essential, however, to show that z satisfies (3.4.1).

Note also that, although (3.4.1) blows-up for $p > 1$, the use of $p > 2$ is crucial in the application of Lemma 2.3.14 in the estimate (2.4.6).

We now use Theorem 1.2.2 to provide an example of LOBC for solutions of (1.2.1)-(1.1.3) in a more general bounded domain. The computations closely follow [61].

Corollary 2.4.2. *Let Ω be a bounded domain satisfying a uniform interior sphere condition. Then, there exist $u_0 \in C^1(\overline{\Omega})$, with $u_0 \geq 0$ and $u_0|_{\partial\Omega} = 0$, such that LOBC occurs for solutions of* (1.2.1)-(1.1.3) *in a finite time $T = T(u_0, \Omega)$.*

Proof: From the interior sphere condition, there exists an $\eta > 0$ such that for all $x_0 \in \partial\Omega$, there exists a ball of radius η tangent to $\partial\Omega$ at x_0, say $B_\eta(x_1)$. Consider $\varphi \in C_0^\infty(B_1(0))$ a radial cut-off function such that

$$\varphi(r) = \begin{cases} 1, & r \leq \frac{2}{3} \\ 0, & r \geq \frac{3}{4} \end{cases}$$

and consider the solution v of

$$\begin{cases} v_t - \mathcal{M}^-(D^2 v) - |Dv|^p = 0 & \text{in } B_1(0) \times (0,\infty), \\ v = 0 & \text{on } \partial B_1(0) \times [0,\infty), \\ v(x,0) = C\varphi(|x|) & \text{in } \overline{B_1(0)}, \end{cases}$$

where the boundary condition is understood in the viscosity sense. It is easy to check that, for large enough $C > 0$, $C\varphi$ satisfies (1.2.2). Hence, by Theorem 1.2.2, LOBC occurs for v at some time $T = T(C\varphi) > 0$. As before, v is radial, thus $v(x, T(C\varphi)) > 0$ for all $x \in \partial B_1(0)$.

We now rescale and translate v to obtain a solution in $B_\eta(x_1)$: define

$$\tilde{v}(x,t) = \eta^k v(\eta^{-1}|x - x_1|, \eta^{-2}t), \tag{2.4.12}$$

where $k = \frac{p-2}{p-1}$. Then $\tilde{v}$ is a solution of (1.2.1) in $B_\eta(x_1) \times (0,\infty)$ satisfying homogeneous boundary data (again in the viscosity sense) and initial condition

$$\tilde{v}(x,0) = C\eta^k \varphi(\eta^{-1}|x - x_1|).$$

We note that, since the rescaling (2.4.12) produces a solution of (1.2.1) on the corresponding rescaled domain, the boundary condition in the viscosity sense is preserved: if the equation holds "up to a boundary point" $(x,t) \in \partial B_1(0) \times (0,\infty)$, it will hold up to the point of $\partial B_\eta(x_1) \times (0,\infty)$ where it is mapped.

The solution $\tilde{v}$ is radially symmetric with respect to x_1. Thus we have $\tilde{v}(x_0, T) > 0$, where $T = \eta^2 T(C\varphi)$.

Define now

$$u_0(x) = \begin{cases} \tilde{v}(x,0) & \text{if } x \in B_\eta(x_1), \\ 0 & \text{if } x \in \overline{\Omega} \backslash B_\eta(x_1), \end{cases}$$

and consider the solution u of

$$\begin{cases} u_t - \mathcal{M}^-(D^2 u) = |Du|^p & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

with u_0 as previously defined. Of course, u is also a solution of (1.2.1) in $B_\eta(x_1) \times (0,\infty)$, and satisfies $u \geq 0$ on $\partial B_\eta(x_1) \times (0,\infty)$ in the viscosity sense. Thus, by comparison we have

$$u \geq \tilde{v} \quad \text{in } \overline{B_\eta(x_1)} \times [0,\infty).$$

Hence,

$$u(x_0,T) \geq \tilde{v}(x_0,T) > 0,$$

i.e., LOBC occurs for u. □

The previous result might be rephrased to include a condition applicable to more general u_0 than the example provided. We avoided this for simplicity, since the condition is rather convoluted, but do so now for completeness.

For any ball $B_\eta(x_1) \subset \Omega$, where $x_1 \in \Omega$, $\eta > 0$ and $\partial B_\eta(x_1)$ is tangent to $\partial\Omega$ at x_0, denote the radial variable by $r = |x - x_1|$. Note $0 < r < \eta$. For any v defined in $B_\eta(x_1)$, we may define the radial symmetrization

$$s(v)(r) = \inf_{\partial B_r(x_1)} v.$$

Note that, for any $u_0 \in C(\overline{\Omega})$ such that $u_0|_{\partial\Omega} = 0$, we have $s(u_0) \leq u_0$ in $B_\eta(x_1)$ and $s(u_0)(\eta) = s(u_0)(x_0) = 0$. Note also that $\|s(u_0)\|_\infty = \|u_0\|_\infty$.

Corollary 2.4.3. *Using the notation above, as well as that of Theorem 1.2.2 and Corollary 2.4.2, there exists positive constants $\delta = \delta(\lambda,\Lambda,N)$ and $M = M(\lambda,\Lambda,N,p)$ such that LOBC occurs for all solutions of* (1.2.1)-(1.1.3) *with initial data u_0 such that*

$$\sup \left\{ \eta^{-k} \int_\delta^{1-\delta} s(u_0)(\eta r)\, dr \right\} > M. \tag{2.4.13}$$

Here, the supremum runs over all $B_\eta(x_1) \subset \Omega$ tangent to $\partial\Omega$ for fixed $\eta > 0$.

Proof: The corresponding proof is analogous to that of Corollary 2.4.2, in that it follows by a comparison argument and scaling between $B_\eta(x_1)$ and $B_1(0)$. Hence, it will be omitted. □

Remark 2.4.4. After a change of variable, condition (2.4.13) can be written as

$$\sup\left\{\int_{\eta\delta}^{\eta(1-\delta)} s(u_0)(r)\,dr\right\} > \eta^{k+1}M,$$

which more closely resembles the condition given for Theorem 1.2.2, in that the limits of integration and the constant on the right-hand side reflect the dependence on Ω (through η), in addition to λ, Λ, N and p. Note that the supremum still runs over all interior tangent spheres $B_\eta(x_1)$ for fixed $\eta > 0$.

2.5 Extensions

To extend the results regarding LOBC to more general equations, we must first guarantee that the global existence result of [10] applies to the equations considered. In fact, that the result applies to our model equation, (1.2.1), will follow as a particular case. For convenience, we restate part of what was mentioned in the introduction. Namely, consider

$$u_t - F(D^2u) = f(Du) \text{ in } \Omega \times (0, T),$$

where the nonlinearities are as follows: $F : S(N) \to \mathbb{R}$ is uniformly elliptic, i.e.,

$$\mathcal{M}^-(X - Y) \le F(X) - F(Y) \le \mathcal{M}^+(X - Y) \quad \text{for all } X, Y \in S(N),$$

and vanishes at zero. In particular, this implies that

$$\mathcal{M}^-(X) \le F(X) \le \mathcal{M}^+(X) \quad \text{for all } X \in S(N). \tag{2.5.1}$$

The gradient nonlinearity $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $f(\xi) \ge |\xi|^2 h(|\xi|)$ for all $\xi \in \mathbb{R}^n$, where $h : \mathbb{R} \to \mathbb{R}$ satisfies the growth condition (1.2.5) and

$$h = h(s) \text{ is positive nondecreasing for } s > 0, \tag{2.5.2}$$

$$s \mapsto s^2 h(s) \text{ is convex}, \tag{2.5.3}$$

$$h(yz) \le C(h(y) + h(z)) \text{ for large } y, z > 0 \text{ and some } C > 0. \tag{2.5.4}$$

The last condition implies that h grows more slowly than any positive power. Examples which satisfy the conditions above are $h(s) = (\log s)^p$ and $h(s) = (\log s)^p(\log\log s)^q$, for large s and $p, q > 0$.

Comparison, existence and uniqueness

We look to verify hypotheses (H1) and (H2) needed for the Strong Comparison Principle, Theorem 2.1.1. We begin by setting

$$G(x, r, \xi, X) = G(\xi, X) = -F(X) + f(\xi), \tag{2.5.5}$$

where $\xi \in \mathbb{R}^N$, $X \in S(N)$, and the nonlinearities F, f (and consequently, h) are as above. Note that this is compatible with the exchange of sub- and supersolutions mentioned in Remark 2.1.3, since $\tilde{F} : S(N) \to \mathbb{R}$ given by

$$\tilde{F}(X) = -F(-X), \qquad \text{for all } X \in S(N)$$

is uniformly elliptic and vanishes at zero if F does.

We proceed to check (i) through (iii) of property (P) for $h_1(s) = s^2 h(s)$, with h as above:

(i) By the growth condition (1.2.5), we have

$$\int_1^\infty \frac{s}{h_1(s)}\, ds = \int_1^\infty \frac{s}{s^2 h(s)} = \int_1^\infty \frac{1}{sh(s)}\, ds < \infty.$$

(ii) Since h is nondecreasing,

$$L \mapsto L^2 s^2 h(Ls) - CL^2 s^2 h(s) = L^2 s^2 (h(Ls) - Ch(s))$$

is increasing for all $s > 0$ and $L \geq 1$.

(iii) Since $L, s > 0$ will be taken large, it is equivalent to show that, for fixed $C, \tilde{C} > 0$ and $\epsilon > 0$,

$$h(Ls) - Ch(s) \geq \epsilon,$$
$$\epsilon > \tilde{C}/Ls.$$

It follows from the growth condition (1.2.5) on h that $h(s) \to \infty$ as $s \to \infty$. Hence, fixing $s > 0$ and taking large enough $L = L(s) > 1$, we get $h(Ls) \geq \epsilon + Ch(s)$. The second inequality above comes from choosing L large as well.

This shows (1.2.3) satisfies (H1). Now on to (H2). The second matrix inequality in (H2) implies $X \leq Y + o(1)$. Hence, $\mu X \leq Y + o(1)$ for any $0 < \mu < 1$, and from the uniform ellipticity and the definition of the Pucci operator, this gives

$$F(\mu X) - F(Y) \leq \mathcal{M}^+(\mu X - Y) \leq o(1).$$

For the contribution of the gradient term to the estimate of (H2), if suffices to have h_1 above (i.e., $s \mapsto s^2 h(s)$) be locally Lipschitz and satisfy the following, as noted in Example 1 of [10]:

(H3) For all $C > 0$, there exists a sequence $0 < \mu_\epsilon < 1$ defined for $0 < \epsilon \leq 1$ such that $\mu_\epsilon \to 1$ as $\epsilon \to 0$ and such that for all large $r > 0$ large enough and $0 < \epsilon$ small enough, we have:

$$C\epsilon r \sup_{0 \leq \tau \leq r(1+C\epsilon)} |h_1'(\tau)| \leq (1 - \mu_\epsilon) \inf_{\tau \geq r(1 - C(1-\mu_\epsilon))} (h_1'(\tau)\tau - h_1(\tau)).$$

Given the properties of h above, a lengthy but straightforward computation shows that to verify (H3) it suffices to choose μ_ϵ such that $\epsilon^{-1}(1 - \mu_\epsilon) \to +\infty$ as $\epsilon \to 0$.

Loss of boundary conditions

What follows is an extension of Theorem 1.2.2 that includes a more general gradient term, with a suitable growth condition. Consider

$$u_t - \mathcal{M}^-(D^2u) = g(|Du|) \quad \text{in} \quad B_1(0) \times (0,T), \tag{2.5.6}$$

where $g : \mathbb{R} \to \mathbb{R}$, g is convex increasing for $s \geq 0$, and $g(0) = 0$. Note that (2.5.6) has no "(x,t)-dependence", so that Lemma 2.3.7 applies directly. Lemma 2.3.1 applies as well if u_0 is radially symmetric, given that $g = g(|Du|)$.

On the other hand, Proposition 2.3.8 requires a slight adaptation. In the case that $w(\hat{x},\hat{t}) < u_{\epsilon,\kappa}(\hat{x},\hat{t})$, where $(\hat{x},\hat{t})$ is a point of second-order differentiability of the regularized function w, we must take the regularization parameter $\delta = \delta(g)$ small enough so that

$$\begin{aligned} w_t(\hat{x},\hat{t}) - \mathcal{M}^-(D^2w(\hat{x},\hat{t})) - g(|Dw(\hat{x},\hat{t})|) \geq& -\frac{K}{\kappa^{\frac{1}{2}}} + \frac{\lambda}{\delta} - \frac{\Lambda(n-1)}{\epsilon} - g\left(\frac{K}{\epsilon^{\frac{1}{2}}}\right) \\ \geq& \; 0 \end{aligned}$$

to get that w is a supersolution.

For the statement of the following Lemma, we define

$$a(s) = \sup_{y>0} \frac{g^{-1}(ys)}{g^{-1}(y)}, \quad s \geq 0,$$

and recall the definition of the convex conjugate,

$$g^*(s) = \sup\{\, ys - g(y) \,|\, y \in \mathbb{R}\,\}.$$

Lemma 2.5.1. *Let $u_0 \in C(\overline{B_1(0)})$ be a radial function, and g as described above. Assume also that g is such that* (2.5.6) *satisfies* (H1) *and* (H2) *of Section 2.1. If*

$$\int_1^\infty \frac{g^*(La(s))}{s^2}\, ds < \infty \tag{2.5.7}$$

for all $L > 0$, then there exist positive constants δ and M, depending only on λ, Λ, N and g, such that, if

$$\int_\delta^{1-\delta} u_0(r) - \frac{1}{2}\|u_0\|_\infty \, dr > M, \tag{2.5.8}$$

then the solution u of (2.5.6), (1.1.2), (1.1.3) *with $\Omega = B_1(0)$ and initial data u_0 has LOBC at some finite time $T = T(u_0)$.*

Proof: Aside from using all the auxiliary results leading to Theorem 1.2.2, the proof follows that of Theorem 5.2 in [57]. We repeat most of the argument for convenience. Once more, we proceed by contradiction, assuming u is a solution which satisfies (1.1.2) in the classical sense. We consider φ_1 as previously defined, and again denote by w the function obtained by regularizing the radial part of the solution u of (2.5.6) for $\Omega = B_1(0)$. This function now satisfies, for arbitrary $\epsilon > 0$ and $0 < t_0 < t_1 < T$,

$$\tilde{\rho} w_t \geq (\rho w')' + \tilde{\rho} g(|w'|) \quad \text{for a.e. } r \in (\epsilon, 1-\epsilon), t \in (t_0, t_1).$$

From the definition of a, setting $y = g(|w'(r,t)|)\varphi_1(r,t)$ for $r \in (\epsilon, 1-\epsilon)$, $t_0 < t < t_1$, we have

$$a(1/\varphi_1) g^{-1}(g(|w'|)\varphi_1) \geq g^{-1}(y/\varphi_1) = |w'|.$$

Hence,

$$g(|w'|)\varphi_1 \geq g\left(\frac{|w'|}{a(1/\varphi_1)}\right). \tag{2.5.9}$$

Let $L > 0$ to be chosen later. By the definition of the convex conjugate, we have

$$L|w'| = \frac{|w'|}{a(1/\varphi_1)} La(1/\varphi_1) \leq g\left(\frac{|w'|}{a(1/\varphi_1)}\right) + g^*(La(1/\varphi_1))$$

for a.e. $r \in (\epsilon, 1-\epsilon), t_0 < t < t_1$ (we have omitted the arguments for simplicity). This is an instance of Fenchel's inequality, analogous to that of Hölder's inequality in the proof of Theorem 1.2.2.

Using (2.5.9), we have

$$L \int_\epsilon^{1-\epsilon} |w'|\tilde{\rho}\, dr \leq \int_\epsilon^{1-\epsilon} g(|w'|)\varphi_1 \tilde{\rho}\, dr + \int_\epsilon^{1-\epsilon} g^*(La(1/\varphi_1))\tilde{\rho}\, dr.$$

That the second integral is finite follows from (2.5.7). It can also be proven that it is bounded by a constant C that does not depend on ϵ, as in Lemma 2.3.14.

Arguing as in the beginning of the proof of Theorem 1.2.2, we obtain

$$\dot{z}(t) + \lambda_1 z(t) \geq I$$

where z is defined exactly as before, but now

$$\begin{aligned} I &:= \int_\epsilon^{1-\epsilon} g(|w'|)\varphi_1 \tilde{\rho}\, dr \\ &\geq L \int_\epsilon^{1-\epsilon} |w'|\, \tilde{\rho}\, dr - \int_\epsilon^{1-\epsilon} g^*(La(1/\varphi_1))\tilde{\rho}\, dr \\ &\geq L \int_\epsilon^{1-\epsilon} |w'|\, \tilde{\rho}\, dr - C \\ &\geq L \left(\int_\epsilon^{1-\epsilon} |w|\, \tilde{\rho}\, dr - w(1-\epsilon, t)\right) - C, \end{aligned}$$

where the last inequality is follows by applying Lemma 2.3.16. Setting L sufficiently large, in terms of $\|\varphi_1\|_\infty$ and λ_1 (both of which are independent of ϵ), and noting that $w(1-\epsilon, t) = o(1)$ as $\epsilon \to 0$, as before, we obtain

$$\dot{z}(t) \geq z(t) - C \quad \text{for a.e. } t \in (t_0, t_1), \tag{2.5.10}$$

which we can integrate to get

$$z(t) \geq (z(t_0) - C)e^{t-t_0} \quad \text{for all } t \in (t_0, t_1). \tag{2.5.11}$$

To conclude, note that (2.5.10) does not blowup in finite time, as does (3.4.1). We will find a contradiction in the form of a bound, derived from (2.5.11), which we can easily violate by choosing the appropriate u_0. Also, since our aim is to prove nonexistence beyond some finite time, we may assume $T > 0$ is large to achieve this contradiction. By choosing the time-regularization parameter small as well, the difference $t_1 - t_0$ can be made large as well, say $t_1 - t_0 \geq T/2$.

As in (2.4.11), we have

$$z(t_0) \geq \int_\delta^{1-\delta} u_0(r)\hat{\varphi}(r)\hat{\rho}(r)\, dr + o(1), \quad \text{as } \epsilon,\, t_0 \to 0. \tag{2.5.12}$$

On the other hand, by evaluating (2.5.11) at $t = t_1$,

$$\begin{aligned} z(t_0) &\leq e^{-(t_1-t_0)} z(t) + C \\ &\leq e^{-\frac{T}{2}} \int_\epsilon^{1-\epsilon} w(r,t)\varphi_1(r)\tilde{\rho}\, dr + C. \end{aligned}$$

Recalling the bounds for φ_1, $\tilde{\rho}$, that $\hat{\varphi}$, $\hat{\rho}$ are bounded by below by a positive constant in $[\delta, 1-\delta]$, and that $\|w\|_\infty \leq \|u_0\|_\infty$, we take T sufficiently large so that

$$z(t_0) \leq \frac{1}{2}\int_\delta^{1-\delta} \|u_0\|_\infty \hat{\varphi}\hat{\rho}\, dr + C.$$

Combining both estimates for $z(t_0)$, we have

$$\int_\delta^{1-\delta} (u_0 - \frac{1}{2}\|u_0\|_\infty)\hat{\varphi}(r)\hat{\rho}(r)\, dr < C,$$

where $C = C(\lambda, \Lambda, N, g)$. This bound is readily violated by choosing a suitably large u_0, and as before, it is equivalent to the one in the statement of the lemma since $\hat{\varphi}$ and $\hat{\rho}$ are uniformly bounded from below in $[\delta, 1-\delta]$. □

Finally, we proceed with the proof of the extension mentioned in the introduction. *Proof of Theorem 1.2.3:* First, define $g(s) = s^2 h(s)$, where h satisfies (2.5.2)-(2.5.4) and the growth condition (1.2.5). It is proven in Lemma 5.3 and the Completion of Theorem 2.2 in [57] that g so defined satisfies the hypothesis of Lemma 2.5.1, including (2.5.7). Furthermore, using (2.5.1), if u is a solution of (1.2.3), formally we have that

$$u_t - \mathcal{M}^-(D^2u) \geq u_t - F(D^2u) = f(Du) \geq |Du|^2 h(|Du|) = g(|Du|) \quad \text{in } \Omega \times (0,T),$$

and this is readily checked using test functions. Hence, u is a supersolution of (2.5.6) in $\Omega \times (0,T)$. Using the interior sphere condition, without loss of generality we may assume that $B_1(0) \subset \Omega$ with $B_1(0)$ tangent to $\partial\Omega$ at some point $x_0 \in \partial\Omega \cap \partial B_1(0)$ (This is equivalent to repeating the constructions of Theorems 1.2.2 and 2.5.1 on a ball of arbitrary radius and performing translation.) Arguing as in the proof of Corollary 2.4.2, we conclude that u is a supersolution of (2.5.6) in $B_1(0) \times (0,T)$. Let $\tilde{u}_0 \in C(\overline{B_1(0)})$ nonnegative, radially symmetric and satisfies (2.5.8), and consider the solution $\tilde{u}$ of (2.5.6) with initial data $\tilde{u}_0$ and homogeneous boundary data. By Lemma 2.5.1, $\tilde{u}$ has LOBC in finite time $T' > 0$, and since it is radially symmetric we may conclude that LOBC occurs at $x_0 \in B_1(0)$. Now define

$$u_0(x) = \begin{cases} \tilde{u}_0(x), & x \in B_1(0) \\ 0, & x \in \overline{\Omega} \backslash B_1(0). \end{cases}$$

By comparison, we have that the solution u of (1.2.3)-(1.1.2)-(1.1.3) with initial data u_0 satisfies $u(x_0, T') \geq \tilde{u}(x_0, T') > 0$, hence u has LOBC. □

Remark 2.5.2. The more general nonlinearities do not admit a rescaling argument like the one given in the proof of Corollary 2.4.2. Furthermore, (1.2.3) may no longer have radial symmetry. The preceding argument is in some sense simpler, but it does not provide a condition one can check for any given initial data like the one in Corollary 2.4.3.

Remark 2.5.3. The typical example for the nonlinearity h in Theorem 1.2.3 is $h(s) = (\log s)^q$ for $q > 0$ and large s. In this case, the growth condition (1.2.5) forces that $q > 1$. This is consistent with what is known to be a more precise condition for preventing GBU in the case of the viscous Hamilton-Jacobi equation: for

$$u_t - \Delta u = f(u, \nabla u) \quad \text{in } \Omega \times (0,T),$$

GBU does not occur if

$$|f(u, \nabla u)| \leq C(u)(1 + |\nabla u|^2) h(|\nabla u|)$$

where $C(u)$ is locally bounded, and h is positive nondecreasing and satisfies

$$\int_1^\infty \frac{1}{s h(s)} = \infty.$$

See [51], Ch. IV, and the references therein.

Remark 2.5.4. The proof of Theorem 1.2.3 uses only the uniform interior sphere condition. Both interior and exterior sphere conditions are assumed to establish a connection between Theorems 1.2.1 and 1.2.3. Specifically, to have both results applicable in the same situation. See also Remarks 2.2.2.

Chapter 3

Analysis of the attainment of boundary conditions for a nonlocal diffusive Hamilton-Jacobi equation[1]

The Chapter is organized as follows. In Sec. 3.1 we recall the notion of solution from [20], which is used throughout our work, and provide some remarks on the relevant results contained in that work. Sec. 3.2 is devoted to the proof of Theorem 1.2.4. In Sec. 3.3 we provide the technical results required for the proof of Theorem 1.2.5, which is proved in Sec. 3.4.

3.1 Notion of solution

We recall the notion of solution for (1.2.6) given in [20]: let $\delta > 0$, $\phi \in C^2(B_\delta(x) \times (t-\delta, t+\delta))$ and $v \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ a bounded measurable function. Define

$$E_\delta(v, \phi, x, t) = \phi_t(x,t) + I[B_\delta(x)](\phi(\cdot,t)) + I[\mathbb{R}^N \setminus B_\delta(x)](v(\cdot,t)) - |D\phi(x,t)|^p, \tag{3.1.1}$$

where, for a measurable $A \subset \mathbb{R}^N$ and a bounded, measurable function ψ,

$$I[A](\psi(\cdot,t)) = C_{N,s}\, P.V. \int_A \frac{\psi(x,t) - \psi(y,t)}{|x-y|^s}\, dy.$$

We momentarily consider a non-homogeneous boundary condition

$$u(x,t) = g(x) \quad \text{for all } (x,t) \in \mathbb{R}^N \setminus \overline{\Omega} \times [0,T], \tag{3.1.2}$$

with $g \in C_b(\mathbb{R}^N \setminus \Omega)$, and define the upper (resp., lower) g-extension of $u \in USC(\overline{\Omega} \times [0,T])$ (resp., $v \in LSC(\overline{\Omega} \times [0,T])$) as the function defined in $\mathbb{R}^N \times \mathbb{R}$

[1]This chapter is based on the article [48].

as

$$u^g(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega \times [0,T], \\ g(x,t) & \text{if } (x,t) \in \mathbb{R}^N \setminus \overline{\Omega} \times [0,T], \\ \max\{u(x,t), g(x,t)\} & \text{if } (x,t) \in \partial\Omega \times [0,T] \end{cases}$$

(resp.,

$$v_g(x,t) = \begin{cases} v(x,t) & \text{if } (x,t) \in \Omega \times [0,T], \\ g(x,t) & \text{if } (x,t) \in \mathbb{R}^N \setminus \overline{\Omega} \times [0,T], \\ \min\{v(x,t), g(x,t)\} & \text{if } (x,t) \in \partial\Omega \times [0,T]). \end{cases} \tag{3.1.3}$$

Definition 3.1.1. *A function* $u \in USC(\overline{\Omega} \times [0,T])$ (resp., $v \in LSC(\overline{\Omega} \times [0,T])$) *is a subsolution* (resp. supersolution) *of* (1.2.6)-(3.1.2)-(1.1.3) *if for every* $\phi \in C^2(B_\delta(x) \times (t-\delta, t+\delta))$ *with* $\delta > 0$, *and every maximum* (resp. minimum) *point* (x_0,t_0) *of* $u^g - \phi$ (resp. $v_g - \phi$) *over* $B_\delta(x) \times (t-\delta, t+\delta)$, *we have*

$$\begin{aligned} E_\delta(u^g, \phi, x_0, t_0) \le 0 & \quad \text{if } (x_0,t_0) \in \Omega \times (0,T], \\ \min\{E_\delta(u^g, \phi, x_0, t_0), u(x_0,t_0) - g(x_0,t_0)\} \le 0 & \quad \text{if } (x_0,t_0) \in \partial\Omega \times (0,T], \\ \min\{E_\delta(u^g, \phi, x_0, t_0), u(x_0,t_0) - u_0(x_0,t_0)\} \le 0 & \quad \text{if } x_0 \in \overline{\Omega},\ t_0 = 0, \end{aligned}$$

where E_δ *is defined as in* (3.1.1). (resp.

$$\begin{aligned} E_\delta(v_g, \phi, x_0, t_0) \ge 0 & \quad \text{if } (x_0,t_0) \in \Omega \times (0,T], \\ \max\{E_\delta(v_g, \phi, x_0, t_0), v(x_0,t_0) - g(x_0,t_0)\} \ge 0 & \quad \text{if } (x_0,t_0) \in \partial\Omega \times (0,T], \\ \max\{E_\delta(v_g, \phi, x_0, t_0), v(x_0,t_0) - u_0(x_0,t_0)\} \ge 0 & \quad \text{if } x_0 \in \overline{\Omega},\ t_0 = 0.) \end{aligned}$$

A viscosity solution of (1.2.6)-(3.1.2)-(1.1.3) *is a function whose upper and lower semicontinuous envelopes are respectively sub- and supersolutions, as defined above. LOBC is said to occur whenever the "classical" inequality corresponding to* (3.1.2) *fails at some point, e.g., if for a subsolution* u *we have*

$$u(x_0,t_0) > g(x_0) \quad \text{for some } (x_0,t_0) \in \partial\Omega \times (0,T].$$

In this case the generalized condition implies that $E_\delta(u^g, \phi, x_0, t_0) \le 0$ *for any* ϕ *as above.*

Remark 3.1.2. To be precise, the equation to which the results of [20] apply is

$$u_t + (-\Delta)^s u + |Du|^p = 0 \quad \text{in } \Omega \times (0,T), \tag{3.1.4}$$

which differs from (1.2.6) only in the sign of the nonlinearity (here Ω, T, s, and p are as before). This does not in any way complicate the analysis, since a simple sign change allows us to go from one equation to the other; i.e., if u is a subsolution (resp. supersolution) of (3.1.4), then $\tilde{u} = -u$ is a supersolution (resp. subsolution) of (1.2.6).

Remark 3.1.3. Definition 3.1.1 also interprets the initial condition (1.1.3) in the viscosity sense, given that $\overline{\Omega} \times \{t = 0\}$ (the "bottom" of the domain) is part of the parabolic boundary of $\Omega \times (0,T)$. However, as noted in [20], Lemma 4.1, there is no LOBC on this set. That is, if u and v are respectively a bounded, upper-semicontinuous subsolution and a bounded, lower-semicontinuous supersolution of (1.2.6)-(1.2.7)-(1.1.3), then

$$u(x,0) \leq u_0(x) \leq v(x,0) \quad \text{for all } x \in \overline{\Omega}.$$

Similarly, as a consequence of [20], Proposition 4.3, and Remark 3.1.2 above, there is no LOBC for supersolutions of (1.2.6)-(1.2.7)-(1.1.3) either. More precisely, if v is a bounded, lower-semicontinuous supersolution of (1.2.6)-(1.2.7)-(1.1.3), then

$$v(x,t) \geq 0 \quad \text{for all } x \in \partial\Omega,\ t \in [0,T].$$

Remark 3.1.4. An important consequence of the comparison result of [20] is that solutions of (1.2.6)-(1.2.7)-(1.1.3) are uniformly bounded for all $0 \leq t \leq T$. Indeed, from the assumptions on the initial data, we have that $\underline{v} \equiv 0$ and

$$\bar{v}(x) = \begin{cases} \|u_0\|_\infty & \text{if } x \in \overline{\Omega}, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash \overline{\Omega}. \end{cases}$$

are respectively sub- and supersolutions to (1.2.6)-(1.2.7)-(1.1.3). Hence, by comparison $0 \leq u(x,t) \leq \|u_0\|_\infty$ for all $(x,t) \in \overline{\Omega} \times (0,T)$.

3.2 Local existence

From the discussion in the Introduction and from Remark 3.1.3, Theorem 1.2.4 follows if we can construct a suitable *supersolution* satisfying (1.2.7) in the classical sense. To this end we follow the corresponding construction in [27], which addresses a similar (stationary) problem. For convenience, we state the key estimates obtained therein.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, C^2 domain and $s \in (0,1)$. Then, there exists a $\delta > 0$ such that, for each $0 < \alpha < s$ there exists $c_1 > 0$ such that*

$$(-\Delta)^s (d(x)^\alpha) \geq c_1 d(x)^{\alpha - 2s} \quad \textit{for all } x \in \Omega_\delta.$$

The constant c_1 depends on N, s and α, and is such that $c_1 \to 0$ as $\alpha \to s^-$.

Proof. This is a special case of Lemma 3.1 in [27]. □

Lemma 3.2.2. *Let $\alpha \in (0,2s)$ and let $y \in \mathbb{R}^N$. Then, there exists a constant $c_2 > 0$ such that*

$$(-\Delta)^s (|\cdot - y|^\alpha) \geq -c_2 |x-y|^{\alpha - 2s} \quad \textit{for all } x \in \mathbb{R}^N \backslash \{y\}.$$

Moreover, there exists a constant $\bar{c}_2 > 0$ such that $c_2 \leq \bar{c}_2$ as $s \to 1^-$.

Proof. This is a special case of Lemma 3.3 in [27]. □

Proof of Theorem 1.2.4: Let $u = u(x,t)$ denote the viscosity solution of (1.2.6)-(1.2.7)-(1.1.3), which a priori satisfies (1.2.7) only in the viscosity sense. Our aim is to construct a function $\bar{v} = \bar{v}(x)$ that satisfies the following:

$$(-\Delta)^s \bar{v} \geq |D\bar{v}|^p \quad \text{in } \Omega_\delta \times (0, T^*), \tag{3.2.1}$$

$$\bar{v} = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega \times (0, T^*), \tag{3.2.2}$$

$$\bar{v} \geq u \quad \text{in } \Omega \backslash \Omega_\delta \times (0, T^*), \tag{3.2.3}$$

$$\bar{v} \geq u_0 \quad \text{in } \overline{\Omega}_\delta, \tag{3.2.4}$$

where $\delta > 0$ and $T^* > 0$ are yet to be determined. Here (3.2.2), (3.2.3) and (3.2.4) are meant in the pointwise sense. Note that $\bar{v}$ is time-independent; time only plays a role in (3.2.3), which can fail for a sufficiently large time.

Let $y \in \partial\Omega$, $\beta^* < \alpha < \min\{\beta, s\}$, and let $\lambda, \mu > 0$ to be chosen later. We write $M = [u_0]_{\beta,\overline{\Omega}}$ and define

$$v_y(x) = \lambda M |x-y|^\alpha + \mu M d(x)^\alpha. \tag{3.2.5}$$

Note that v_y satisfies

$$v_y(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N \backslash \Omega$$

and

$$v_y(y) = 0. \tag{3.2.6}$$

For all $x \in \Omega_\delta$, denote by $\pi(x)$ the unique point of $\partial\Omega$ such that $d(x) = |x - \pi(x)|$; $\pi(x)$ is well defined for small enough $\delta > 0$ and smooth $\partial\Omega$. Assume $\delta \leq 1$ and $\mu \geq 1$. Using (1.2.11) and $\alpha < \beta$, we have, for all $x \in \overline{\Omega}_\delta$,

$$\begin{aligned} v_y(x) &\geq \mu M d(x)^\alpha = \mu [u_0]_{\beta,\overline{\Omega}} |x - \pi(x)|^\alpha \geq [u_0]_{\beta,\overline{\Omega}} |x - \pi(x)|^\beta \\ &\geq |u_0(x) - u_0(\pi(x))| = u_0(x), \end{aligned}$$

where we have used that $u_0 \geq 0$ and $u_0(\pi(x)) = 0$, by (1.2.11). This shows that v_y satisfies (3.2.4).

Now, set $\lambda > 0$ large enough so that

$$\lambda |x-y|^\alpha > |x-y|^\beta \quad \text{for all } x \in \Omega \backslash \Omega_\delta. \tag{3.2.7}$$

Thus, reasoning as above,

$$v_y(x) > \lambda M |x-y|^\alpha \geq [u_0]_\beta |x-y|^\beta \geq u_0(x) \quad \text{for all } x \in \Omega \backslash \Omega_\delta.$$

Note this inequality is *strict* since we are at a positive distance from the boundary. Since $v_y - u_0$ is strictly positive over the compact set $\Omega \backslash \Omega_\delta$, there exists $\epsilon > 0$ such that

$$u_0(x) + \epsilon < v_y(x) \quad \text{for all } x \in \Omega \backslash \Omega_\delta.$$

Recall that the continuous viscosity solution u satisfies (1.1.3) in the classical sense (see Remark 3.1.3). This implies that $u(\cdot, t) \to u_0$ as $t \to 0^+$ uniformly over $\overline{\Omega}$. Therefore, there exists $T^* > 0$ such that

$$u(x,t) < u_0(x) + \epsilon < v_y(x), \quad \text{for all } x \in \Omega \backslash \Omega_\delta \text{ and all } t < T^*.$$

This gives (3.2.3) for v_y.

Applying Lemmas 3.2.1 and 3.2.2, and using that $d(x) \le |x-y|$, we have

$$\begin{aligned}(-\Delta)^s v_y(x) &\ge \mu M c_1 d(x)^{\alpha-2s} - \lambda M c_2 |x-y|^{\alpha-2s} \\ &\ge M d(x)^{\alpha-2s}(\mu c_1 - \lambda c_2) \qquad \text{for all } x \in \Omega_\delta,\end{aligned}$$

for all sufficiently small $\delta > 0$. On the other hand, using that $\alpha < 1$ and again that $d(x) \le |x-y|$, we compute

$$|Dv_y(x)|^p \le M^p d(x)^{p(\alpha-1)}(\mu+\lambda)^p.$$

Combining these estimates, we obtain

$$\begin{aligned}&(-\Delta)^s(v_y(x)) - |Dv_y(x)|^p \\ &\quad \ge M d(x)^{\alpha-2s}\left(\mu c_1 - \lambda c_2 - M^{p-1}(\mu+\lambda)^p d(x)^{p(\alpha-1)-(\alpha-2s)}\right).\end{aligned}$$

We now take $\mu > 0$ large enough, so that $\mu c_1 - \lambda c_2 > \frac{\mu c_1}{2}$, then take $\delta > 0$ small enough, so that

$$M^{p-1}(\mu+\lambda)^p \delta^{p(\alpha-1)-(\alpha-2s)} < \frac{\mu c_1}{4}. \tag{3.2.8}$$

Thus,

$$(-\Delta)^s v_y(x) - |Dv_y(x)|^p \ge M d(x)^{\alpha-2s}\left(\frac{\mu c_1}{4}\right) \ge 0,$$

which gives that v_y satisfies (3.2.1).

By standard arguments, the function

$$v(x) = \inf_{y\in\partial\Omega} v_y(x)$$

is a viscosity supersolution of (3.2.1). It also satisfies (3.2.3) and (3.2.4), since these are satisfied by v_y for all $y \in \partial\Omega$. Furthermore, v is continuous across $\partial\Omega$ and, by (3.2.6), satisfies

$$v(x) \le v_x(x) = 0 \quad \text{for all } x \in \partial\Omega.$$

Therefore, applying the comparison principle of [20] over the domain $\Omega_\delta \times (0,T^*)$ we obtain that $u(x,t) \le v(x)$ for all $x \in \overline{\Omega}_\delta$, $0 \le t \le T^*$. Hence we conclude that

$$u(x,t) \le v(x) = 0 \quad \text{for all } x \in \partial\Omega,\ 0 \le t \le T^*.$$

□

Remark 3.2.3. Local existence can be proven for initial data with "critical" regularity, i.e., $u_0 \in C^{\beta^*}(\overline{\Omega})$, $\beta^* = \frac{p-2s}{p-1}$, in exactly the same way, provided $[u_0]_{\beta^*,\overline{\Omega}}$ is sufficiently small. More precisely, we define

$$v_y(x) = M|x-y|^{\beta^*} + \mu M d(x)^{\beta^*},$$

with $M = [u_0]_{\beta^*,\overline{\Omega}}$. Proceeding as above, we set $\mu > 0$ so that $\mu c_1 - c_2 > \frac{\mu c_1}{2}$, and require that

$$[u_0]_{\beta^*,\overline{\Omega}} \leq \left(\frac{\mu c_1}{2(\mu+1)^p} \right)^{\frac{1}{p-1}}$$

is satisfied, instead of (3.2.8). Note that the estimate corresponding to (3.2.7) is satisfied automatically.

Remark 3.2.4. The estimates from Lemmas 3.2.1 and 3.2.2 are stable as $s \to 1^-$. Therefore, Theorem 1.2.4 implies the existence of local solutions for the Dirichlet problem associated to (1.1.1) with $u_0 \in C^\beta(\overline{\Omega})$, with $\beta > \beta^* = \frac{p-2}{p-1}$; and, following Remark 3.2.3, for $u_0 \in C^{\beta^*}(\overline{\Omega})$, provided $[u_0]_{\beta^*,\overline{\Omega}}$ is sufficiently small.

3.3 Technical results

Regularization

In this section we use the regularization by inf-sup-convolution once more to obtain a supersolution of (1.2.6) which has the regularity needed for the proof of Theorem 1.2.5 (see Section 2.3 for the relevant definitions and properties). The supersolution obtained converges to the viscosity solution u of (1.2.6) uniformly over $\overline{\Omega} \times [0,T]$ for any $T > 0$ as the regularization parameters tend to zero.

For a given $v \in C(\overline{\Omega} \times [0,T])$, we will obtain a function which is Lipschitz continuous with respect to t and $C^{1,1}$ with respect to x, following [24].

First, denote by $\tilde{v}$ the lower "0-extension" of v, as defined in (3.1.3). That is, $\tilde{v} = v_g$ with $g \equiv 0$ (this is only to avoid the notation "v_0"). For $v \in C(\overline{\Omega} \times [0,T])$ with $v|_{\partial\Omega} \equiv 0$, this leads to $\tilde{v} \in BUC(\mathbb{R}^N \times [0,T])$. We remark that this is precisely what a solution of (1.2.6)-(1.2.7)-(1.1.3) with no LOBC satisfies. We then iterate the convolution operators defined above:

$$w := ((\tilde{v}_{\epsilon,\kappa})_\delta)^\delta = (\tilde{v}_{\epsilon+\delta,\kappa})^\delta, \tag{3.3.1}$$

where $\epsilon, \delta, \kappa > 0$. The expression furthest to the right follows from Proposition 2.3.5, (vii).

As a first step we recall that inf-convolution (resp., sup-convolution) preserves supersolutions (resp. subsolutions) in the viscosity sense, albeit in a proper subset of the original domain.

Lemma 3.3.1. *Let $u \in C(\overline{\Omega} \times [0,T])$ be the (unique) viscosity solution of* (1.2.6)*. Then $\tilde{u}_{\epsilon,\kappa}$ (see Definition 2.3.4), is a viscosity supersolution of* (1.2.6) *in $\Omega^{\epsilon^*} \times (\kappa^*, T-\kappa^*]$, where*

$$\epsilon^* = 2\sqrt{\epsilon\|u\|_\infty}, \quad \kappa^* = 2\sqrt{\kappa\|u\|_\infty}, \tag{3.3.2}$$

and $\|u\|_\infty = \sup_{\Omega\times(0,T)} u$.

Proof. This is a time-dependent version of Proposition 5.5 in [22], in the particular case where the equation in its entirety is translation invariant, i.e., when there is no "(x,t)-dependence". In this case, the regularized function satisfies exactly the same inequality as the original supersolution. □

Remark 3.3.2. We briefly explain the origin of the constants appearing in Lemma 3.3.1. The proof uses the fact that the infimum in Definition 2.3.4 is achieved at some $(\hat{y},\hat{s}) \in \mathbb{R}^N \times [0,T]$. A simple computation then shows that $|\hat{y}-x| \le \epsilon^*$, $|\hat{s}-t| \le \kappa^*$, as defined in Lemma 3.3.1. Therefore, to ensure that $(\hat{y},\hat{s}) \in \Omega \times (0,T)$, so that we can test the equation at this point, we require that $d(x) > \epsilon^*$ and $\kappa^* < t < T-\kappa^*$.

We also remark that, although slightly different from the one given in Definition 3.1.1, the notion of solution given in [22] is essentially the same when concerning the behavior of either sub- and supersolutions at interior points (in particular, for the purposes of Lemma 3.3.1).

We now state a key proposition concerning the eigenvalues of a regularized function.

Proposition 3.3.3. *Let $v \in BUC(\mathbb{R}^N)$, $\delta > 0$ and suppose that $w = (v_\delta)^\delta$ is differentiable everywhere. If for some $\hat{x}$, $w(\hat{x}) < v(\hat{x})$ and w is twice differentiable at $\hat{x}$, then $D^2 w(\hat{x})$ has $-\frac{1}{\delta}$ as an eigenvalue.*

Proof. This is Proposition 4.4 in [24], save for the order in which the inf- and sup-convolutions are performed. The proof is entirely analogous. □

Proposition 3.3.4. *Let $\eta > 0$, $0 < t_0 < t_1 < T$, and u be a bounded, lower-semicontinuous supersolution of (1.2.6) in $\Omega \times (0,T)$. Then there exist $\epsilon, \delta, \kappa > 0$ such that $w = (\tilde{u}_{\epsilon+\delta,\kappa})^\delta$, as defined above, satisfies*

$$w_t + (-\Delta)^s w - |Dw|^p \ge 0 \quad \text{a.e. } in\ \Omega^\eta \times (t_0,t_1). \tag{3.3.3}$$

Proof. Let ϵ and κ be respectively so small that $\epsilon^* < \eta$ and $\kappa^* < \min\{t_0, T-t_1\}$ (see (3.3.2)). This implies that $\Omega^\eta \subset \Omega^{\epsilon^*}$ and $(t_0,t_1) \subset (\kappa^*, T-\kappa^*)$. Applying Lemma 3.3.1, we have that $\tilde{u}_{\epsilon,\kappa}$ is a viscosity supersolution of (1.2.6) in $\Omega^\eta \times (t_0,t_1)$. From Proposition 2.3.5 (viii), we know that $w \le \tilde{u}_{\epsilon,\kappa}$ in $\mathbb{R}^N \times [0,T]$, and from Proposition 2.3.5 (v) and (ix), that w has a second order "parabolic" Taylor expansion a.e. in $\Omega^\eta \times (t_0,t_1)$. Let $(\hat{x},\hat{t}) \in \Omega^\eta \times (t_0,t_1)$ be any point where such an expansion exists.

Assume first that $w(\hat{x},\hat{t}) = \tilde{u}_{\epsilon,\kappa}(\hat{x},\hat{t})$. In this case, Proposition 2.3.5 (v) implies that

$$\phi(x,t) = w(\hat{x},\hat{t}) + w_t(\hat{x},\hat{t})(t-\hat{t}) + \langle Dw(\hat{x},\hat{t}), x-\hat{x}\rangle + \frac{1}{2}\langle D^2 w(\hat{x},\hat{t})(x-\hat{x}), x-\hat{x}\rangle$$

satisfies $\phi(x,t) \le \tilde{u}_{\epsilon,\kappa}$ over $B_\gamma(\hat{x}) \times (\hat{t}-\gamma, \hat{t}+\gamma)$ for some $\gamma > 0$. As $\phi \in C^2$, it is a valid test function in the sense of Definition 3.1.1 (see this Definition for the notation that follows). Since $\tilde{u}_{\epsilon,\kappa}$ is a viscosity supersolution, this implies $E_\gamma(\tilde{u}_{\epsilon,\kappa}, \phi, \hat{x}, \hat{t}) \ge 0$. As w is $C^{1,1}$ with respect to x, $I[B_\gamma(\hat{x})](w(\hat{x},\hat{t}))$

is classically defined. Moreover, the error from the second-order expansion results in a summable term over B_γ, hence

$$I[B_\gamma(\hat{x})](w(\hat{x},\hat{t})) = I[B_\gamma(\hat{x})](\phi(\hat{x},\hat{t})) + o(1) \text{ as } \gamma \to 0. \tag{3.3.4}$$

On the other hand, $w \le \tilde{u}_{\epsilon,\kappa}$ implies that

$$I[\mathbb{R}^N \backslash B_\gamma(\hat{x})](w(\hat{x},\hat{t})) \ge I[\mathbb{R}^N \backslash B_\gamma(\hat{x})](\tilde{u}_{\epsilon,\kappa}(\hat{x},\hat{t})).$$

Putting this together, we have

$$w_t(\hat{x},\hat{t}) + (-\Delta)^s w(\hat{x},\hat{t}) - |Dw(\hat{x},\hat{t})|^p + o(1) \ge E_\gamma(\tilde{u}_{\epsilon,\kappa}, \phi, \hat{x}, \hat{t}) \ge 0.$$

We thus obtain (3.3.3) at $(\hat{x},\hat{t})$ by taking $\gamma \to 0$.

Assume now that $w(\hat{x},\hat{t}) < \tilde{u}_{\epsilon,\kappa}(\hat{x},\hat{t})$. In this case, by Proposition 3.3.3, $D^2 w(\hat{x},\hat{t})$ has an eigenvalue equal to $-\frac{1}{\delta}$. Denoting the eigenvalues of $D^2 w(\hat{x},\hat{t})$ by e_i, $i = 1, \dots, N$, without loss of generality we may write $e_1 = -\frac{1}{\delta}$. Recall from Proposition 2.3.5, (vi) and (ix), that $e_i \le \frac{1}{\epsilon}$ for $i = 2, \dots, N$. Since $D^2 w(\hat{x},\hat{t})$ is symmetric, there exists an orthogonal matrix A such that $D^2 w(\hat{x},\hat{t}) = A^T \operatorname{diag}(e_1, \dots, e_N) A$. Therefore, writing $y = Az$, we have

$$\begin{aligned}\langle D^2 w(\hat{x},\hat{t}) z, z\rangle &= \langle A^T \operatorname{diag}(e_1,\dots,e_N) Az, z\rangle = \langle \operatorname{diag}(e_1,\dots,e_N)Az, Az\rangle \\ &= \sum_{i=1}^N e_i\, y_i^2 \le -\frac{1}{\delta} y_1^2 + \sum_{i=2}^N \frac{1}{\epsilon} y_i^2.\end{aligned}$$

We then use an alternative form of the fractional Laplacian (see, e.g., [29]) and the second-oder expansion at $\hat{x}$ to compute, for a sufficiently small $\gamma > 0$,

$$\begin{aligned}(-\Delta)^s w(\hat{x},\hat{t}) &= -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{w(\hat{x}+z,\hat{t}) + w(\hat{x}-z,\hat{t}) - 2w(\hat{x},\hat{t})}{|z|^{N+2s}}\, dz \\ &= -\frac{C_{N,s}}{2} \int_{B_\gamma(0)} \frac{\langle D^2 w(\hat{x},\hat{t})z, z\rangle + o(|z|^2)}{|z|^{N+2s}}\, dz \\ &\quad - \frac{C_{N,s}}{2} \int_{\mathbb{R}^N \backslash B_\gamma(0)} \frac{w(\hat{x}+z,\hat{t}) + w(\hat{x}-z,\hat{t}) - 2w(\hat{x},\hat{t})}{|z|^{N+2s}}\, dz \\ &=: I_1 + I_2.\end{aligned}$$

The uniform convergence of the regularization (Proposition 2.3.5 (iv)) we have, taking sufficiently small regularization parameters, that $\|w\|_\infty \le \|u\|_\infty + 1$, hence

$$\begin{aligned}I_2 &\ge -2C_{N,s}\|w\|_\infty \int_{\mathbb{R}^N \backslash B_\gamma(0)} \frac{1}{|z|^{N+2s}}\, dz \\ &\ge -2C_{N,s}(\|u\|_\infty + 1) \int_{\mathbb{R}^N} \frac{1}{|z|^{N+2s}}\, dz =: -C_0.\end{aligned}$$

For I_1, the error term from the expansion is controlled as in (3.3.4). Using also that the change of variables $y = Az$ is orthogonal, we compute

$$\begin{aligned} I_1 &\geq -\frac{C_{N,s}}{2}\int_{B_\gamma(0)} \frac{\langle D^2 w(\hat{x},\hat{t})z, z\rangle}{|z|^{N+2s}}\,dz - 1 \\ &\geq \frac{C_{N,s}}{2}\left(\frac{1}{\delta}\int_{B_\gamma(0)} \frac{y_1^2}{|y|^{N+2s}}\,dy - \frac{1}{\epsilon}\sum_{i=2}^{N}\int_{B_\gamma(0)} \frac{y_i^2}{|y|^{N+2s}}\,dy\right) - 1 \end{aligned}$$

A standard computation shows that, for $i = 1, \ldots, N$,

$$0 < \int_{\mathbb{R}^N} \frac{y_i^2}{|y|^{N+2s}}\,dy =: C_1 < \infty.$$

Thus, combining the above estimates we have

$$(-\Delta)^s w(\hat{x},\hat{t}) \geq \frac{C_{N,s}C_1}{2}\left(\frac{1}{\delta} - \frac{N-1}{\epsilon}\right) - C_0 - 1$$

Recalling the bounds for w_t and Dw given in Proposition 2.3.5 (iii), we finally obtain

$$\begin{aligned} &w_t(\hat{x},\hat{t}) + (-\Delta)^s w(\hat{x},\hat{t}) - |Dw(\hat{x},\hat{t})|^p \\ &\quad\geq -\frac{K}{\kappa^{\frac{1}{2}}} + \frac{C_{N,s}C_1}{2}\left(\frac{1}{\delta} - \frac{N-1}{\epsilon}\right) - \frac{K^p}{\epsilon^{\frac{p}{2}}} - C_0 - 1. \end{aligned}$$

Therefore, taking δ sufficiently small with respect to κ and ϵ, the right-hand side of the above inequality becomes nonnegative. Hence, we obtain (3.3.3) at $(\hat{x},\hat{t})$ once more, and the result follows. □

Remark 3.3.5. The first part of the proof of Proposition 3.3.4, i.e., working under the assumption $w(\hat{x},\hat{t}) = \tilde{u}_{\epsilon,\kappa}(\hat{x},\hat{t})$, amounts to showing the equivalence of using punctually-$C^{1,1}$ test functions instead of C^2 in Definition 3.1.1. See Lemma 4.3 in [22] for the time-independent case.

The principal eigenvalue problem for the Dirichlet fractional Laplacian

In this section we provide some results regarding the principal eigenvalue problem for the fractional Laplacian on domains approximating Ω, i.e.,

$$\begin{cases} (-\Delta)^s\varphi = \lambda\varphi & \text{in } \Omega^\eta, \\ \varphi = 0 & \text{in } \mathbb{R}^N\backslash\Omega^\eta. \end{cases} \tag{3.3.5}$$

The existence of a solution pair $(\lambda_1^\eta, \varphi_1^\eta)$ of (3.3.5) where $\lambda_1^\eta > 0$, and φ_1^η is nonnegative in Ω^η and unique up to a multiplicative constant is proved in [54], Proposition 9. The solution obtained in this work is in the weak, or variational, sense. In particular, $\varphi_1^\eta \in H^s(\Omega^\eta)$. Furthermore, in [53], Proposition 4, it is proved that $\varphi_1^\eta \in L^\infty(\Omega^\eta)$. We set

$$\|\varphi_1^\eta\|_\infty = 1 \quad \text{for all } \eta > 0. \tag{3.3.6}$$

From here, it is possible to apply the results of [52] to obtain that $\varphi_1^\eta \in C^s(\mathbb{R}^N)$. Once the "right-hand side" of (3.3.5) is continuous, the notions of weak and viscosity solution coincide (see [52], Remark 2.11). Moreover, "bootstrapping" the results contained in [52] (see also [22]), the solution can be shown to be regular enough in the interior of Ω^η for (3.3.5) to hold in a classical, pointwise sense.

Additionally, since $\partial\Omega^\eta$ is smooth (see Remark 3.3.6), it can be shown that as a consequence of Hopf's lemma and the strong maximum principle ([37]) that

$$\lambda_1^\eta = \sup\{\lambda > 0 \mid \exists\, \varphi > 0 \text{ in } \Omega^\eta \text{ such that } (-\Delta)^s\varphi \geq \lambda\varphi\}. \tag{3.3.7}$$

Remark 3.3.6. Towards the proof of our main result, our aim is to provide estimates that remain uniform with respect to the varying domain (i.e, independent of η). To this end, we recall a few basic facts concerning the geometry of the domains Ω^η, $\eta > 0$. First, since Ω is C^2 by assumption, there exists an $\eta_0 > 0$ such that the distance function $d|_{\Omega\setminus\overline{\Omega}^{2\eta_0}}$ is C^2; in particular Ω^η is C^2 for all $\eta \in (0, 2\eta_0)$.

The following is a classical result from the geometry of hypersurfaces.

Proposition 3.3.7. *Let κ_i and κ_i^η, $i = 1, \dots, N-1$, denote the principal curvatures of $\partial\Omega$ at $y_0 \in \partial\Omega$ and of $\partial\Omega^\eta$ at $y = y_0 + \eta\nu(y_0)$, respectively. Then*

$$\kappa_i^\eta = \frac{\kappa_i}{1 - \eta\kappa_i}. \tag{3.3.8}$$

Proof. See, e.g., [38], Chap. 2. □

Remark 3.3.8. In particular, a C^2 domain satisfies an exterior uniform sphere condition. Furthermore, at any given point of the boundary, the radius of the exterior tangent sphere is bounded by below by the smallest of the radii of curvature, which are equal to the inverses of the principal curvatures. Proposition 3.3.7 allows us to extend the uniform exterior sphere condition to domains close to Ω *in a uniform way.* More precisely, there exists a positive constant ρ_0, depending only on Ω and η_0, as given by Remark 3.3.6, such that for all $\eta \in (0, \eta_0)$, and for all $y \in \partial\Omega^\eta$, there exists $y_1 \in \mathbb{R}^N \setminus \Omega^\eta$ such that $\overline{B_{\rho_0}(y_1)} \cap \overline{\Omega^\eta} = \{y\}$.

Stability of eigenfunctions

Theorem 3.3.9. *Let η_0 be as in Remark 3.3.6. Then, there exists C depending only on Ω, N, and s, such that, for all $\eta \in (0, \eta_0)$, the positive solution of (3.3.5), normalized as above, satisfies*

$$\|\varphi_1^\eta\|_{C^s(\mathbb{R}^N)} \leq C.$$

Proof. Theorem 3.3.9 follows readily from the corresponding estimate for the Dirichlet problem. Indeed, we apply Proposition 1.1 from [52] to the solution of (3.3.5), recalling the normalization (3.3.6), and obtain

$$\|\varphi_1^\eta\|_{C^s(\mathbb{R}^N)} \leq C_2 \|\lambda_1^\eta \varphi_1^\eta\|_{L^\infty(\mathbb{R}^N)} = C_2 \lambda_1^\eta \leq C_2 \lambda_1^{\eta_0} \tag{3.3.9}$$

for some positive C_2 depending on N, s, and Ω^η. For the last inequality we used the fact that $\Omega^{\eta_0} \subset \Omega^\eta$ for $\eta < \eta_0$, and therefore $\lambda_1^\eta \leq \lambda_1^{\eta_0}$, by (3.3.7).

It remains only to verify that the constant $C_2 = C_2(N, s, \Omega^\eta)$ in (3.3.9) (i.e., the constant in Proposition 1.1 from [52]) can be taken uniformly for $\eta \in (0, \eta_0)$, with η_0 fixed, from Remark 3.3.6. We perform this analysis in Appendix 3.A. □

Theorem 3.3.10 (Stability of eigenvalues)**.** *Let λ_1 and φ_1 denote the principal eigenvalue for $(-\Delta)^s$ and the associated eigenfunction with $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_\infty = 1$, respectively. Then, as $\eta \to 0$, $\lambda_1^\eta \to \lambda_1$ and $\varphi_1^\eta \to \varphi_1$ uniformly in $\overline{\Omega}$.*

Proof. By (3.3.7), we have that $\lambda_1^\eta \to \hat{\lambda}$ for some $\hat{\lambda} \geq \lambda_1$. Using Theorem 3.3.9 and (3.3.6), by compactness we have that $\varphi_1^\eta \to \hat{\varphi}$ for some $\hat{\varphi} \in C(\mathbb{R}^N)$, locally uniformly in $\mathbb{R}^N$. In particular, $\varphi_1^\eta \to \hat{\varphi}$ uniformly in $\overline{\Omega}$, and $\hat{\varphi} \equiv 0$ in $\mathbb{R}^N \backslash \Omega$. Also, we have $\hat{\varphi} \geq 0$ and, from (3.3.6), that $\|\hat{\varphi}\|_\infty = 1$. In particular, $\hat{\varphi} \not\equiv 0$.

Let $U \subset\subset \Omega$ and $\eta' > 0$ small enough so that $U \subset \Omega^{\eta'}$ (hence, by (3.3.7), $U \subset\subset \Omega^\eta$ for all $\eta < \eta'$). Then the equation in (3.3.5) is satisfied in U for all $\eta < \eta'$. Together with the considerations of the preceding paragraph, by the stability of viscosity solutions (see, e.g., Corollary 4.6 in [22]), we have that $(-\Delta)^s \hat{\varphi} = \hat{\lambda} \hat{\varphi}$ in U. Furthermore, by the strong maximum principle, we have that $\hat{\varphi} > 0$ in U. Since the choice of U was arbitrary, the above argument implies that

$$\begin{cases} (-\Delta)^s \hat{\varphi} = \hat{\lambda} \hat{\varphi} & \text{in } \Omega, \\ \hat{\varphi} = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

By (3.3.7), this implies $\lambda_1 \geq \hat{\lambda}$. Hence, $\hat{\lambda} = \lambda_1$, and, by uniqueness, $\hat{\varphi} = \varphi_1$. □

Remark 3.3.11. The following is a consequence of Lemma 3.3.10 that will be useful later on: given $K \subset\subset \Omega$ and $\eta' > 0$ small enough, there exists a positive constant c, depending on K and η', such that, for all $\eta \in (0, \eta')$, $\varphi_1^\eta(x) > c$ for all $x \in K$.

A uniform Hopf's lemma

Lemma 3.3.12. *Let η_0 be as in Remark 3.3.6. Then, there exists C_3, depending only on N, s, and Ω, such that, for all $\eta \in (0, \eta_0)$,*

$$(-\Delta)^s d_\eta(x)^s = f_\eta(x) \quad \textit{for all } x \in \Omega,\ \eta < d(x) < 2\eta_0, \tag{3.3.10}$$

for some $f_\eta \in L^\infty(\Omega^\eta \backslash \overline{\Omega}^{2\eta_0})$ with $\|f_\eta\|_\infty \leq C_3$.

For a fixed domain, this is contained in Lemma 3.9 in [52]. For completeness, we go into the details of the proof of this result to show that the right-hand side of (3.3.10) is uniformly bounded. To this end, we also use elements from the proof of the corresponding result for the (more general) case of the fractional p-Laplacian in [39] (Theorem 3.6). Additionally, we employ Proposition 3.3.7.

Proof of Lemma 3.3.12. Fix $\eta > 0$ and $y \in \partial\Omega^\eta$. Through a covering argument it suffices to obtain (3.3.10) in a set $(\Omega^\eta \backslash \overline{\Omega}^{2\eta_0}) \cap B_\rho(y)$ for some $\rho > 0$. Since $\partial\Omega^\eta$ is C^2 for small enough $\eta > 0$, there exists a change of variables $\Psi^\eta \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, $\Psi^\eta(X) = x$, such that $\Psi^\eta = I$ in $\mathbb{R}^N \backslash B_\rho(y)$ and

$$d_\eta(x) = d_\eta(\Psi^\eta(X)) = X_N, \quad \text{for all } x \in (\Omega^\eta \backslash \overline{\Omega}^{2\eta_0}) \cap B_\rho(y). \tag{3.3.11}$$

From this change of variables we define $v_\eta(x) := ((\Psi^\eta)^{-1}(x) \cdot e_N)_+^s = d_\eta(x)^s$ and obtain (3.3.10), with

$$\|f_\eta\|_\infty \leq C(\|D\Psi^\eta\|_\infty, \|(D\Psi^\eta)^{-1}\|_\infty, \rho)\|D^2\Psi^\eta\|_\infty. \tag{3.3.12}$$

By further rotation and translation we can assume $y = \Psi^\eta(0)$,

$$\Psi_i^\eta(X) = X_i, \quad \text{for } i = 1, \dots, N-1, \tag{3.3.13}$$

and set up a principal coordinate system at $y \in \partial\Omega^\eta$ (see, e.g., [36], Sec. 14.6). In these coordinates, we have (using the notation of Lemma 3.3.7)

$$D\Psi^\eta(x) = \operatorname{diag}[1 - d_\eta(x)\kappa_1^\eta, \dots, 1 - d_\eta(x)\kappa_{N-1}^\eta, 1] \tag{3.3.14}$$

for $x = y + d_\eta(x)\nu^\eta(y)$. This implies that

$$\|D\Psi^\eta\|_\infty, \|D(\Psi^\eta)^{-1}\|_\infty \leq C(\kappa_1^\eta, \dots, \kappa_{N-1}^\eta). \tag{3.3.15}$$

From (3.3.13), we have $\frac{\partial^2 \Psi_k^\eta}{\partial X_i \partial X_j} = 0$ for $k \neq N$; $i, j = 1, \dots, N$. Implicit differentiation of (3.3.11) yields

$$\begin{aligned} &\frac{\partial^2 d}{\partial x_j \partial x_N}(\Psi^\eta(X)) \frac{\partial \Psi_N^\eta}{\partial X_i} + \frac{\partial^2 d}{\partial x_N^2}(\Psi^\eta(X)) \frac{\partial \Psi_N^\eta}{\partial X_j} \frac{\partial \Psi_N^\eta}{\partial X_i} + \frac{\partial d}{\partial x_n}(\Psi^\eta(X)) \frac{\partial^2 \Psi_N^\eta}{\partial X_j \partial X_i} = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = 1, \dots, N;\ j = 1, \dots, N-1; \\ &\frac{\partial^2 d}{\partial x_N^2}(\Psi^\eta(X)) \frac{\partial \Psi_N^\eta}{\partial X_N} \frac{\partial \Psi_N^\eta}{\partial X_i} + \frac{\partial d}{\partial x_n}(\Psi^\eta(X)) \frac{\partial^2 \Psi_N^\eta}{\partial X_N^2} = 0, \quad \text{for } i = 1, \dots, N. \end{aligned} \tag{3.3.16}$$

We also know, for x as in (3.3.14), that

$$D^2 d(x) = \operatorname{diag}\left[\frac{\kappa_1^\eta}{1 - d_\eta(x)\kappa_1^\eta}, \dots, \frac{\kappa_{N-1}^\eta}{1 - d_\eta(x)\kappa_{N-1}^\eta}, 1\right]. \tag{3.3.17}$$

Since $\frac{\partial d}{\partial x_n}(y) = \frac{\partial d}{\partial x_n}(\Psi^\eta(0)) = 1$, by continuity we have that

$$\frac{\partial d}{\partial x_n}(x) = \frac{\partial d}{\partial x_n}(\Psi^\eta(X)) \neq 0 \quad \text{for all } x \in B_\rho(y).$$

This, together with (3.3.15), (3.3.16) and (3.3.17), implies that

$$\|D^2\Psi^\eta\|_\infty \le C(\kappa_1^\eta,\ldots,\kappa_{N-1}^\eta).$$

Combining the above estimates with (3.3.12) and Proposition 3.3.7, we conclude that $\|f_\eta\|_\infty$ is uniformly bounded by a constant that depends only on N, s, and Ω; more specifically, on N, s, and the the principal curvatures of $\partial\Omega$. □

The following computation is adapted from [28].

Lemma 3.3.13 (Uniform Hopf's Lemma). *Let η_0 be as in Remark 3.3.6. Then, there exists a positive constant c_3, depending only on N, s, and Ω, such that for all $\eta \in (0,\eta_0)$, the solution of (3.3.5) satisfies*

$$\varphi_1^\eta(x) \ge c_3 d_\eta(x)^s \quad \text{for all } x\in\Omega,\ \eta \le d(x) < \eta_0.$$

Proof. Write $K_0 = \overline{\Omega}^{\eta_0}$ for short and define, for $A > 0$,

$$v(x) = d_\eta(x)^s + A\chi_{K_0}(x),$$

where χ_{K_0} denotes the characteristic function of K_0. Note that $v \in USC(\mathbb{R}^N)$ and $v \in C(\mathbb{R}^N\backslash K_0)$. From Lemma 3.3.12, for $x \in \Omega\cap\{\eta < d(x) < \eta_0\}$ we have

$$(-\Delta)^s v(x) = f_\eta(x) + A\,C_{N,s}\,P.V.\int_{\mathbb{R}^N} \frac{-\chi_{K_0}(y)}{|x-y|^{N+2s}}\,dy \le C_3 - AC(K_0),$$

where C_3 is the constant from Lemma 3.3.12 and $C(K_0) > 0$. Hence, for sufficiently large A, $(-\Delta)^s v(x) \le 0$.

As a consequence of Theorem 3.3.10, φ_1^η is bounded by below on compact sets, uniformly in η (see Remark 3.3.11). We therefore take $c_3 > 0$ small enough so that

$$c_3(\mathrm{diam}(\Omega)^s + A) \le \inf_{K_0} \varphi_1^\eta. \tag{3.3.18}$$

Then, for all $\eta \in (0,\eta_0)$,

$$(-\Delta)^s(c_3 v) \le 0 \le \lambda_1^\eta\varphi_1^\eta \quad \text{in } \Omega\cap\{\eta < d(x) < \eta_0\}, \tag{3.3.19}$$

and

$$c_3 v = c_3(d_\eta^s + A\chi_{K_0}) \le c_3(\mathrm{diam}(\Omega)^s + A) \le \inf_{K_0}\varphi_1^\eta \le \varphi_1^\eta \quad \text{in } \overline{\Omega}^{\eta_0}. \tag{3.3.20}$$

Since $v \equiv \varphi_1^\eta \equiv 0$ in $\mathbb{R}^n\backslash\Omega^\eta$, by comparison we have that

$$c_3 v \le \varphi_1^\eta \quad \text{in } \Omega\cap\{\eta \le d(x) < \eta_0\}.$$

We conclude by observing that $v \equiv d_\eta(\cdot)^s$ in $\mathbb{R}^N\backslash\overline{\Omega}^{\eta_0}$. □

Lemma 3.3.14. *Let η_0 be as in Remark 3.3.6. Then, there exists a positive constant C_4, depending only on $\Omega, N, s,$ and p, such that, for all $\eta \in (0, \eta_0)$,*

$$\int_{\Omega^\eta} (\varphi_1^\eta)^{-\frac{1}{p-1}} \, dx \leq C_4.$$

Proof. By Lemma 3.3.12, we have, for η_0 and c_3 as before,

$$\varphi_1^\eta(x)^{-\frac{1}{p-1}} \leq (c_3 d_\eta(x)^s)^{-\frac{1}{p-1}} \leq C d_\eta(x)^{-\frac{s}{p-1}} \quad \text{for all } x \in \Omega^\eta \backslash \Omega^{\eta_0}.$$

Using the change of variables $\Psi^\eta \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ introduced in Lemma 3.3.12, together with a covering argument and the estimates provided therein, we have

$$\int_{\Omega^\eta \backslash \Omega^{\eta_0}} d_\eta^{-\frac{s}{p-1}} \, dx = C \int_\eta^{\eta_0 - \eta} X_N^{-\frac{s}{p-1}} dX_N \leq C \int_0^{\eta_0} X_N^{-\frac{s}{p-1}} \, dX_N.$$

By assumption (1.2.9), $-\frac{s}{p-1} > -1$, hence this last integral is finite.

On the other hand, using that $\Omega^{\eta_0} \subset\subset \Omega$, together with Lemma 3.3.10 and Remark 3.3.11, we have $\varphi_1^\eta(x) > c_4 > 0$ for all $x \in \Omega^{\eta_0}$, where c_4 depends only on Ω, N and s (through η_0). Hence,

$$\int_{\Omega^\eta} (\varphi_1^\eta)^{-\frac{1}{p-1}} \, dx \leq C \int_0^{\eta_0} X_N^{-\frac{s}{p-1}} \, dX_N + \int_{\Omega^{\eta_0}} c_4^{-\frac{1}{p-1}} \, dx =: C_4.$$

□

3.4 Nonexistence of global solutions and LOBC

As in our previous work [49], the proof of Theorem 1.2.5 uses key ideas from that of Theorem 2.1 in [57]. For completeness, we reproduce some of the elements of [49]. We remark that some care is required in choosing certain parameters appearing in our argument in the correct order, a difficulty which is not present in [57]. Specifically, we first choose u_0 large in an appropriate sense, then take η (which depends on u_0 and the regularization parameters of Sec. 3.3) sufficiently small.

Proof of Theorem 1.2.5. Consider the differential inequality

$$\begin{cases} \dot{y}(t) \geq C y(t)^p, & 0 < t_0 < t < t_1, \\ y(t_0) = M_0, \end{cases} \tag{3.4.1}$$

where C, $M_0 > 0$. We can integrate (3.4.1) explicitly to obtain

$$0 \leq y(t)^{1-p} \leq C(1-p)(t - t_0) + M_0^{1-p}.$$

Hence, $y(t)^{1-p} \to 0$ as $t \to t_0 + \frac{M_0^{1-p}}{C(p-1)}$. Since $1-p < 0$, this implies $y(t) \to +\infty$. Alternatively, for a fixed $t_1 > t_0$, blow-up occurs for $t < t_1$ provided we have

$$M_0 > [C(p-1)(t_1 - t_0)]^{-\frac{1}{p-1}}. \tag{3.4.2}$$

So fix $T > 0$ and assume that the viscosity solution u of (1.2.6) satisfies (1.2.7) in the classical sense. We will later specify the largeness condition on u_0 in terms of M_0 above, but may consider it set from now on, since it depends only on constants already available. The constant C in (3.4.1) is also specified later, depending only on the appropriate quantities. In particular, it is independent of both η and u_0.

Recall the approximate equation obtained in Proposition 3.3.4,

$$w_t + (-\Delta)^s w - |Dw|^p \geq 0 \quad \text{a.e. in } \Omega^\eta \times (t_0, t_1),$$

where now w is obtained by regularization of the viscosity solution u of (1.2.6). Define $z(t) = \int_{\Omega^\eta} w(x,t)\varphi_1^\eta(x)\,dx$, where φ_1^η is the unique positive solution of (3.3.5) normalized so that $\|\varphi_1^\eta\|_\infty = 1$. From Proposition 2.3.5 (iv) and Remark 3.1.4, we obtain, for sufficiently small η, that $\|w\|_\infty \leq \|u\|_\infty + 1 \leq \|u_0\|_\infty + 1$. Thus $z(t)$ is uniformly bounded for $0 \leq t \leq T$. In what remains of the proof, we show that z satisfies (3.4.1) by using the assumption that the solution u satisfies (1.2.7) in the classical sense, and hence blows-up, a contradiction.

From (3.3.3), we compute

$$\begin{aligned}\dot{z}(t) &= \int_{\Omega^\eta} w_t(x,t)\varphi_1^\eta(x)dx \geq \int_{\Omega^\eta} \left[-(-\Delta)^s w(x,t) + |Dw(x,t)|^p\right]\varphi_1^\eta(x)\,dx \\ &= -\int_{\Omega^\eta} (-\Delta)^s w(x,t)\varphi_1^\eta(x)\,dx + \int_{\Omega^\eta} |Dw(x,t)|^p \varphi_1^\eta(x)\,dx =: I_1 + I_2.\end{aligned}$$

We proceed to "integrate by parts" in I_1, taking some care to handle the *P.V.* in the definition of the fractional Laplacian.

Since $\varphi_1^\eta \equiv 0$ in $\mathbb{R}^N \backslash \Omega^\eta$ (point-wise), we have

$$\begin{aligned}I_1 &= -\int_{\Omega^\eta} (-\Delta)^s w(x,t)\varphi_1^\eta(x)\,dx = -\int_{\mathbb{R}^N} (-\Delta)^s w(x,t)\varphi_1^\eta(x)\,dx \\ &= -C_{N,s}\int_{\mathbb{R}^N} \lim_{\omega\to 0}\int_{\mathbb{R}^N\backslash B_\omega(x)} \frac{w(x,t)-w(y,t)}{|x-y|^{N+2s}}\varphi_1^\eta(x)\,dy\,dx.\end{aligned}$$

Since $w(\cdot,t) \in C^{1,1}(\mathbb{R}^N)$ for all $t \in [t_0,t_1]$, $(-\Delta)^s w(\cdot,t)$ is classically defined a.e. in Ω, precisely at the points where w has a second order expansion. Moreover, at such points we have that the integral with respect to y converges as $\omega \to 0$ and, by the standard computation,

$$\left|\int_{\mathbb{R}^N\backslash B_\omega(x)} \frac{w(x,t)-w(y,t)}{|x-y|^{N+2s}}\varphi_1^\eta(x)\,dy\right| \leq C_{N,s}\|D^2 w\|_\infty \varphi_1^\eta(x).$$

Hence by the dominated convergence theorem,

$$I_1 = -C_{N,s}\lim_{\omega\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N\backslash B_\omega(x)} \frac{w(x,t)-w(y,t)}{|x-y|^{N+2s}}\varphi_1^\eta(x)\,dy\,dx.$$

We note that the integrand is no longer singular, and write dV for the measure on $\mathbb{R}^{2N}$. Applying first Fubini's theorem, and then by symmetry (i.e., interchanging x and y), we have

$$\begin{aligned}
I_1 &= -C_{N,s} \lim_{\omega \to 0} \int_{\{|x-y| \geq \omega\}} \frac{w(x,t) - w(y,t)}{|x-y|^{N+2s}} \varphi_1^\eta(x)\, dV \\
&= -C_{N,s} \lim_{\omega \to 0} \int_{\{|x-y| \geq \omega\}} \frac{w(x,t) - w(y,t)}{|x-y|^{N+2s}} (-\varphi_1^\eta(y))\, dV \\
&= -\frac{C_{N,s}}{2} \lim_{\omega \to 0} \int_{\{|x-y| \geq \omega\}} \frac{(w(x,t) - w(y,t))(\varphi_1^\eta(x) - \varphi_1^\eta(y))}{|x-y|^{N+2s}}\, dV.
\end{aligned}$$

Starting from the last integral, we repeat the above computation "in reverse" to pass the operator onto φ_1^η and use the associated eigenvalue problem (3.3.5). Since now $w \equiv 0$ in $\mathbb{R}^N \backslash \Omega$, this gives

$$\begin{aligned}
I_1 &= -\frac{C_{N,s}}{2} \lim_{\omega \to 0} \int_{\{|x-y| \geq \omega\}} \frac{(w(x,t) - w(y,t))(\varphi_1^\eta(x) - \varphi_1^\eta(y))}{|x-y|^{N+2s}}\, dV \\
&= -\int_{\mathbb{R}^N} w(x,t)(-\Delta)^s \varphi_1^\eta(x)\, dx = -\int_{\Omega} w(x,t)(-\Delta)^s \varphi_1^\eta(x)\, dx \\
&= -\int_{\Omega^\eta} w(x,t)(-\Delta)^s \varphi_1^\eta(x)\, dx - \int_{\Omega \backslash \Omega^\eta} w(x,t)(-\Delta)^s \varphi_1^\eta(x)\, dx \\
&= -\lambda_1^\eta z(t) - \int_{\Omega \backslash \Omega^\eta} w(x,t)(-\Delta)^s \varphi_1^\eta(x)\, dx.
\end{aligned}$$

Using that w, $\varphi_1^\eta \geq 0$, and again that $\varphi_1^\eta \equiv 0$ in $\mathbb{R}^N \backslash \Omega^\eta$,

$$\begin{aligned}
&\int_{\Omega \backslash \Omega^\eta} w(x,t)(-\Delta)^s \varphi_1^\eta(x)\, dx = -\int_{\Omega \backslash \Omega^\eta} w(x,t) \int_{\mathbb{R}^N} \frac{\varphi_1^\eta(x) - \varphi_1^\eta(y)}{|x-y|^s}\, dy\, dx \\
&\quad = \int_{\Omega \backslash \Omega^\eta} w(x,t) \int_{\Omega^\eta} \frac{-\varphi_1^\eta(y)}{|x-y|^s}\, dy\, dx = -\int_{\Omega \backslash \Omega^\eta} \int_{\Omega^\eta} \frac{w(x,t)\, \varphi_1^\eta(y)}{|x-y|^s}\, dy\, dx \leq 0.
\end{aligned}$$

Therefore,

$$I_1 \geq -\lambda_1^\eta z(t). \tag{3.4.3}$$

We turn to estimating I_2. First, applying Hölder's inequality and Lemma 3.3.14,

$$\begin{aligned}
\int_{\Omega^\eta} |Dw|\, dx &= \int_{\Omega^\eta} |Dw| (\varphi_1^\eta)^{\frac{1}{p}} (\varphi_1^\eta)^{-\frac{1}{p}}\, dx \\
&\leq \left(\int_{\Omega^\eta} |Dw|^p\, \varphi_1^\eta\, dx \right)^{\frac{1}{p}} \left(\int_{\Omega^\eta} (\varphi_1^\eta)^{-\frac{1}{p-1}}\, dx \right)^{\frac{p-1}{p}} \leq C_4^{\frac{p-1}{p}}\, I_2^{\frac{1}{p}},
\end{aligned} \tag{3.4.4}$$

where C_4 is the constant from Lemma 3.3.14.

We apply the following version of Poincaré's inequality to $w(\cdot,t) \in C^{1,1}(\Omega^\eta)$:

$$\int_{\Omega^\eta} |w(\cdot,t)|\, dx \leq C(\Omega) \left(\sup_{\partial\Omega^\eta} |w(\cdot,t)| + \int_{\Omega^\eta} |Dw(\cdot,t)|\, dx \right), \tag{3.4.5}$$

where the constant appearing on the right-hand side can be given in terms of the diameter of the domain, and can therefore be taken uniformly with respect to η.

From the uniform convergence of the approximation $w \to u$ (Proposition 2.3.5 (iv)) and the uniform continuity of u in $\overline{\Omega} \times [0, T]$, we have that $\sup_{\partial\Omega^\eta} |w(\cdot, t)| \to 0$ as $\eta \to 0$, uniformly in t. Indeed, let $\nu > 0$. The assumption that u satisfies (1.2.7) point-wise implies in particular that $u(\cdot, t) = 0$ in $\partial\Omega$ for all $t \geq t_0$. Therefore, for any $x \in \partial\Omega^\eta$, if $x_0 \in \partial\Omega$ is such that $d(x, x_0) = d(x, \partial\Omega)$, we have

$$\begin{aligned} w(x,t) = w(x,t) - u(x_0,t) &\leq |w(x,t) - u(x,t)| \\ &+ |u(x,t) - u(x_0,t)| < 2\nu. \end{aligned} \tag{3.4.6}$$

Hence, we later write simply $\sup_{\partial\Omega^\eta} |w(\cdot, t)| = o(1)$ as $\eta \to 0$. Together with the normalization $\|\varphi_1^\eta\|_\infty = 1$ and the elementary inequality $(a+b) \leq 2^{p-1}(a^p + b^p)$, (3.4.4), (3.4.5) and (3.4.6) imply

$$\begin{aligned} |z(t)|^p &\leq \left| \int_{\Omega^\eta} w(x,t)\varphi_1^\eta(x)\,dx \right|^p \leq \left(\|\varphi_1^\eta\|_\infty \int_{\Omega^\eta} |w(x,t)|\,dx \right)^p \\ &\leq C(\Omega) \left(\sup_{\partial\Omega^\eta} |w| + \int_{\Omega^\eta} |Dw|\,dx \right)^p \leq o(1) + C \left(\int_{\Omega^\eta} |Dw|\,dx \right)^p \\ &\leq o(1) + CI_2. \end{aligned}$$

Recalling (3.4.4) and (3.4.3), we obtain

$$\dot{z}(t) \geq -\lambda_1^\eta z(t) + C_5 z(t)^p + o(1), \tag{3.4.7}$$

where the $C_5 > 0$ does not depend on either η or u_0; this is crucial for the next step.

We can reduce (3.4.7) to (3.4.1) as follows. We use that $\varphi_1^\eta \to \varphi_1$ uniformly over $\overline{\Omega}$ as $\eta \to 0$; $w \to u$ uniformly over $\overline{\Omega} \times [0, T]$ as $\eta \to 0$ and t_0, $T - t_1 \to 0$ (see Proposition 2.3.5 (iv) and Proposition 3.3.4); that $u(\cdot, t_0) \to u_0$ as $t_0 \to 0$, and the fact that all these functions are uniformly bounded in η, to conclude

$$\begin{aligned} z(t_0) &= \int_{\Omega^\eta} w(x,t_0)\,\varphi_1^\eta(x)\,dx = \int_\Omega w(x,t_0)\,\varphi_1^\eta(x)\,dx + \int_{\Omega\setminus\Omega^\eta} w(x,t_0)\,\varphi_1^\eta(x)\,dx \\ &= \int_\Omega w(x,t_0)\,\varphi_1^\eta(x)\,dx + o(1) = \int_\Omega u_0(x)\,\varphi_1(x)\,dx + o(1) \quad \text{as } \eta \to 0. \end{aligned}$$

Thus we see that the largeness condition for u_0 given in Theorem 1.2.5 implies that $z(t_0)$ is large as well. On the other hand, assumption (1.2.9) implies in particular that $p > 1$, hence $z(t)^p$ is the dominating term in the right-hand side of (3.4.7). Therefore, taking

$$z(t_0) \geq \max\left\{ M_0, \left(\frac{2\lambda_1}{C_5}\right)^{\frac{1}{p-1}} \right\} + 1,$$

and η sufficiently small ensures that both $\dot{z}(t) \geq \frac{C_5}{2} z(t)^p$ for $t > t_0$ and that $z(t_0) \geq M_0$, which together are equivalent to (3.4.1). This gives the desired contradiction. □

Remark 3.4.1. Given the indirect nature of the preceding proof, we would like to highlight the role played by the main assumptions leading to LOBC: the fact that the gradient term is "dominating" in the equation, i.e., $p > s + 1$, from (1.2.9), is used only in Lemma 3.3.14. On the other hand, the assumption that leads to contradiction, that (1.2.7) is satisfied in the classical sense, is used only in (3.4.6) and in applying Poincaré's inequality.

The case of more general boundary conditions can be treated in exactly the same way as above. Assuming $u = g$ in $\mathbb{R}^N \backslash \Omega \times (0, T)$ is satisfied in the classical sense, with $g \in C_b(\mathbb{R}^N \backslash \Omega \times (0, T))$, we obtain

$$\dot{z}(t) \geq -\lambda_1^\eta z(t) + C z(t)^p - C_6,$$

instead of (3.4.7), where C_6 depends on $\|g\|_{L^\infty(\partial\Omega \times (0,T))}$. From here on, the proof continues as above.

Remark 3.4.2. Assuming higher regularity for the initial data, e.g., $u_0 \in C^2(\overline{\Omega})$ it is possible to obtain estimates for u_t (see, e.g., [60], Proposition 4.1, for an example of this method in the local setting). This allows the application of regularity results available for stationary problems (e.g, those of [20]) to our problem, essentially by treating u_t as a bounded "right-hand side". Global Hölder estimates can then be obtained for the solution of (1.2.6)-(1.2.7)-(1.1.3). In this case, Theorems (1.2.4) and (1.2.5) together imply that for any $\beta > \beta^* = \frac{p-2s}{p-1}$, the Hölder semi-norm of the solution u *blows-up in finite time*, i.e., there exists $T_1 > 0$, depending only on N, Ω, s, p and u_0, such that

$$\lim_{t \to T_1} [u(\cdot, t)]_{\beta, \overline{\Omega}} = \infty.$$

This situation is analogous to that of *gradient blow-up* for (1.1.1).

3.A Appendix: Uniform C^s regularity for the approximate domains

In this Appendix we state a version of results from [52] which concern the regularity of solutions to the Dirichlet problem for the fractional Laplacian. We revisit the corresponding proofs to show that the estimates are uniform with respect to varying domains such as those appearing in (3.3.5). In this way we conclude the analysis postponed in the proof of Theorem 3.3.9.

Let $s \in (0, 1)$, $g \in L^\infty(\Omega^\eta)$ and consider

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega^\eta \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega^\eta, \end{cases} \tag{3.A.1}$$

where Ω is a bounded, C^2 domain, $\Omega^\eta = \{x \in \Omega \,|\, d(x) > \eta\}$, $\eta \in (0, \eta_0)$ and η_0 is given by 3.3.6.

Proposition 3.A.1. *Let u be a solution of* (3.A.1). *Then $u \in C^s(\mathbb{R}^N)$ and*

$$\|u\|_{C^s(\mathbb{R}^N)} \leq C\|g\|_{L^\infty(\Omega^\eta)},$$

where C is a constant depending only on Ω and s. In particular, the constant C can be taken uniformly for $\eta \in (0, \eta_0)$.

This is Proposition 1.1 from [52], save for the dependency of C on the parameter η, which we require to be uniform. To this end, we outline the manner in which this result was obtained. We begin stating the key Lemmas leading up to it.

Lemma 3.A.2. *Assume that $w \in C^\infty(\mathbb{R}^N)$ is a solution of $(-\Delta)^s w = h$ in B_2. Then, for every $\beta \in (0, 2s)$,*

$$\|w\|_{C^\beta(\overline{B_{\frac{1}{2}}})} \leq C\left(\|(1+|x|^{-N-2s}w(x)\|_{L^1(\mathbb{R}^N)} + \|w\|_{L^\infty(B_2} + \|h\|_{\infty(B_2)}\right),$$

where the constant C depends only on N, s and β.

Proof. This is Corollary 2.5 in [52]. ☐

Lemma 3.A.3. *There exists a positive constant C and a radial continuous function $\phi_1 \in H^s_{loc}(\mathbb{R}^N)$ satisfying*

$$\begin{cases} (-\Delta)^s\phi_1 \geq 1 & \text{in } B_4\backslash B_1 \\ \phi_1 \equiv 0 & \text{in } B_1 \\ 0 \leq \phi_1 \leq C(|x|-1)^s & \text{in } B_4\backslash B_1 \\ 1 \leq \phi_1 \leq C & \text{in } \mathbb{R}^N\backslash B_4. \end{cases} \tag{3.A.2}$$

Proof. This is Lemma 2.6 in [52]. ☐

Remark 3.A.4. Lemmas 3.A.2 and 3.A.3 have no dependence on the domains Ω nor Ω^η. Therefore, they apply directly to our setting.

Lemma 3.A.5. *Let Ω^η and g be as above, and let u be the solution of 3.A.1. Then*

$$|u(x)| \leq C\|g\|_{L^\infty(\Omega^\eta)}d_\eta(x)^s \quad \text{for all } x \in \Omega^\eta,$$

where C depends only on Ω and s. In particular, C can be taken uniformly in $\eta \in (0, \eta_0)$.

Lemma 3.A.5 relies on the following result:

Lemma 3.A.6. *Let Ω be a bounded domain and let $g \in L^\infty(\Omega^\eta)$. Let u be the solution of* (3.A.1). *Then*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C(\operatorname{diam}(\Omega^\eta))^{2s}\|g\|_{L^\infty(\Omega^\eta)},$$

where C is a constant depending only on N and s.

Proof. This is Claim 2.8 in [52]. Since $\operatorname{diam}(\Omega^\eta) \leq \operatorname{diam}(\Omega)$, the estimate is uniform for $\eta \in (0, \eta_0)$. □

Proof of Lemma 3.A.5: This is Lemma 2.7 in [52]. For points near $\partial\Omega^\eta$, the estimate is obtained by scaling the supersolution from Lemma 3.A.3 to the annular region $B_{2\rho_0} \backslash B_{\rho_0}$, where B_{ρ_0} is an exterior tangent ball to $\partial\Omega$, and applying comparison. Owing to Remark 3.3.8, the scaling can be done uniformly with respect to $\eta \in (0, \eta_0)$. For the remaining points in Ω^η, Lemma 3.A.6 is employed. □

Lemma 3.A.7. *Let Ω be a bounded domain satisfying the exterior ball condition, $g \in L^\infty(\Omega^\eta)$, and u be the solution of 3.A.1. Then $u \in C^s(\Omega^\eta)$ and for all $x_0 \in \Omega^\eta$,*

$$[u]_{C^s(\overline{B_R(x_0)})} \leq C\|g\|_{L^\infty(\Omega^\eta)}, \tag{3.A.3}$$

where $R = \frac{d(x_0)}{2}$ and C depends only on Ω^η, and s.

Proof. This is a special case of Lemma 2.9 in [52]. Although more intricate than that of the previous results, the proof of this result uses only a scaling of the interior estimate of Lemma 3.A.2 to the ball $B_R(x_0)$, the use of the upper barrier for $\|u\|_{L^\infty(\Omega)}$ obtained in 3.A.5, and a covering argument. As such, the constant C in Lemma 3.A.7 now depends on the measure of Ω as well. This quantity, however, varies continuously for Ω^η with $\eta \in (0, \eta_0)$. □

Proof of Proposition 3.A.1: It remains only to extend the estimate from Lemma 3.A.7 up to the boundary. For this we provide an argument from [39]. Through a covering argument, Lemma 3.A.7 extends to an interior bound on any compact subset of Ω^η. Consider now $x, y \in \Omega^\eta$ such that $0 \leq d_\eta(x)$, $d_\eta(y) \leq \rho$ for a small $\rho > 0$ and, without loss of generality, $d_\eta(y) \leq d_\eta(x)$. There are two possible cases:

a) $2|x - y| \leq d_\eta(x)$. This implies that $y \in B_R(x)$ with $R = \frac{d_\eta(x)}{2}$. Hence, applying Lemma 3.A.7, we obtain $|u(x) - u(y)| \leq C\|g\|_{L^\infty(\Omega^\eta)}|x - y|^s$.

b) $2|x-y| > d_\eta(x) \geq d_\eta(y)$. In this case, we apply Lemma 3.A.5 to compute

$$u(x) - u(y) \leq |u(x)| + |u(y)| \leq C(d_\eta(x)^s + d_\eta(y)^s) \leq C|x - y|^s.$$

□

Chapter 4

Large-time behavior of unbounded solutions of viscous Hamilton-Jacobi equations in $\mathbb{R}^N$[1]

The Chapter is organized as follows. In Section 4.1 we prove the existence and uniqueness of solutions to (1.2.15)-(1.2.16), Theorem 1.2.6, stated in the Introduction. In Section 4.2 we construct special sub- and supersolutions which will serve as comparison functions towards obtaining large-time behavior, and finally, in Section 4.3 we prove a result on the uniform boundedness of solutions to (1.2.15)-(1.2.16), Lemma 4.3.1, as well as our main result, Theorem 1.2.7, stated in the Introduction. We note that, in this Chapter, all solutions, subsolutions and supersolutions are meant in the viscosity sense.

4.1 Existence and uniqueness of solutions

In this first section we prove Theorem 1.2.6, stated in the introduction.

Proof of Theorem 1.2.6. As noted in the Introduction, the Theorem follows from Perron's method (see e.g., [40] or [25]), provided we obtain appropriate sub- and supersolutions of (1.2.15)-(1.2.16) and show that a comparison principle is valid. This is done in Proposition 4.1.1 and Theorem 4.1.4 below, respectively. □

A first existence result

Our first task is to construct a bounded from below sub- and supersolutions of (1.2.15)-(1.2.16). To this end we revisit the well-posedness of the parabolic state-constraints problem on bounded domains. In fact, the radial case will sufficient for our purposes.

[1]This chapter is based on the article [17].

Proposition 4.1.1. *Assume $u_0 \in C(\mathbb{R}^N)$ is bounded from below and $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ is nonnegative. Then, there exist continuous, bounded from below sub- and supersolutions of* (1.2.15)-(1.2.16).

As mentioned earlier, we introduce the parabolic state-constraints on $B_R \times (0,T)$, where $B_R = B_R(0)$, the ball of radius R centered at 0:

$$u_t - \Delta u + |Du|^m = f(x) \quad \text{in } B_R \times (0,\infty), \tag{4.1.1}$$

$$u_t - \Delta u + |Du|^m \geq f(x) \quad \text{in } \partial B_R \times (0,\infty), \tag{4.1.2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \overline{B_R}. \tag{4.1.3}$$

Lemma 4.1.2. *Under the assumptions of Proposition 4.1.1, for every $R > 0$ there exists a unique, continuous solution of* (4.1.1)-(4.1.3)*, denoted u^R. Furthermore, if $R' \geq R$, then $u^{R'}(x) \leq u^R(x)$ for all $x \in \overline{B_R}$.*

Proof of Lemma 4.1.2. Uniqueness of solutions follows from the strong comparison result proven in [10], Theorem 3.1. Existence then follows by Perron's method, provided there exist suitable sub- and supersolutions. Since $f(x) \geq 0$ for all $x \in \mathbb{R}^N$, $u \equiv 0$ is a subsolution of (4.1.1)-(4.1.3) for all $R > 0$. By comparison this implies u^R is nonnegative for all $R > 0$. A supersolution may be constructed using the solution of the corresponding *stationary* state-constraint problem, a treatment of which, in the context of viscosity solutions, is provided in [60] (Theorem 3.1).

We proceed to prove the last part of the statement, noting it agrees with the intuitive notion that the solution satisfying the state-constraint condition is maximal among those satisfying (4.1.1) and (4.1.3). Observe that, for $R' \geq R$, $u^{R'}$ is in particular a subsolution of (4.1.1) in $B_R \times (0,\infty)$, satisfying also (4.1.3). Let $\epsilon > 0$, and let ρ_ϵ denote the standard mollifier in $\mathbb{R}^{N+1}$. Since (4.1.1) is convex in (u, Du), $u^{R'} * \rho_\epsilon$ is a classical subsolution of (4.1.1) in B_R with right-hand side $f * \rho_\epsilon$ (this is Lemma 2.7 in [11]). Define, for small $\delta > 0$, $x \in \overline{B_R}$ and $t \geq 0$,

$$w(x,t) = (u^{R'} * \rho_\epsilon)(x,t) - \delta t.$$

We now show that $u^R \geq w$ in $B_R \times (0,T)$ for any $T > 0$. For all $(x,t) \in B_R \times (0,T]$, we compute, for appropriately chosen ϵ and δ,

$$\begin{aligned} &w_t(x,t) - \Delta w(x,t) + |Dw(x,t)|^m \\ &\quad = (u^{R'} * \rho_\epsilon)_t - \Delta(u^{R'} * \rho_\epsilon)(x,t) + |D(u^{R'} * \rho_\epsilon)(x,t)|^m - \delta \\ &\quad \leq (f * \rho_\epsilon)(x) - \delta < f(x). \end{aligned} \tag{4.1.4}$$

On the other hand, assume that

$$(u^R - w)(x_0,t_0) := \min_{\overline{B_R}\times[0,T]} u^R - w < 0.$$

Since w is smooth, the definition of the state-constraints boundary condition (4.1.2) implies that $x_0 \in B_R$. Also, $u^{R'}(x,0) = u_0(x)$ and $w \to u^{R'}$ as $\epsilon, \delta \to 0$ imply that $t_0 > 0$. Hence, we have the viscosity inequality

$$w_t(x_0,t_0) - \Delta w(x_0,t_0) + |Dw(x_0,t_0)|^m \geq f(x_0),$$

which contradicts (4.1.4). Therefore, $u^R \geq w$ in $\overline{B_R} \times [0,T]$. Taking the limit $\epsilon, \delta \to 0$, then $T \to \infty$, we conclude. □

Proof of Proposition 4.1.1. Since $u \equiv 0$ is a subsolution of (1.2.15), it remains only to construct a locally bounded supersolution on the whole space. We proceed to take a half-relaxed limit, using the solutions u^R from Lemma 4.1.2.

Let

$$v^R(x,t) = \begin{cases} u^R(x,t) & \text{if } |x| \leq R, \\ +\infty & \text{if } |x| > R. \end{cases} \tag{4.1.5}$$

This is trivially a solution of (4.1.1) in B_R. We claim that

$$\underline{v}(x,t) = \liminf_{R\to\infty}{}_* \, u^R(x,t)$$

is a supersolution of (1.2.15)-(1.2.16).

For any $x \in \mathbb{R}^N$, $\underline{v}(x,0) \leq u_0(x)$ since this is satisfied for all u^R, $R \geq |x|$. And, for $t > 0$,

$$\underline{v}(x,t) = \liminf_{R\to\infty}{}_* \, u^R(x,t) \leq u^{R'}(x,t) < +\infty,$$

provided $R' > |x|$, since for $R'' \geq R'$, $u^{R''}(x,t) \leq u^{R'}(x,t)$ by Lemma (4.1.2). Thus, by standard arguments (see e.g, [25]), $\underline{v}$ is a supersolution of (1.2.15), and it is nonnegative since u^R is nonnegative for all $R > 0$. □

Remark 4.1.3. The solution obtained in Theorem 4.1.1 is bounded from below and also locally bounded, i.e., bounded on $B_R \times (0,T)$ for all $R, T > 0$, but it can still occur that $u(x,t) \to \infty$ as $t \to \infty$ for some $x \in \mathbb{R}^N$. This is in contrast to the situation of Lemma 4.3.1.

Comparison

Theorem 4.1.4. *Assume that $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ is bounded from below. Let $u \in USC(\mathbb{R}^N \times (0,T))$, $v \in LSC(\mathbb{R}^N \times (0,T); \mathbb{R}^N \cup \{+\infty\})$, $\mathrm{dom}(v) \neq \emptyset$, be bounded from below sub- and supersolutions of* (1.2.15)*, respectively. If $u(\cdot,0) \leq v(\cdot,0)$ in $\mathbb{R}^N$, then $u \leq v$ in all of $\mathbb{R}^N \times [0,\infty)$.*

Proof. To deal with the difficulty of u and v being unbounded, we define

$$z_1(x,t) = -e^{-u(x,t)}, \qquad z_2(x,t) = -e^{-v(x,t)}.$$

Since u and v are bounded from below, z_1 and z_2 are bounded. Furthermore, we have that z_1 and z_2 are respectively a sub- and supersolution of

$$z_t - \Delta z + N(x,z,Dz) = 0, \tag{4.1.6}$$

where $N : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$N(x,r,p) = r\left(f(x) + \left|\frac{p}{r}\right|^2 - \left|\frac{p}{r}\right|^m\right). \tag{4.1.7}$$

Formally, this is a straightforward computation. Given the monotonicity of the transformation defining z_1, z_2, there is no difficulty in passing the computation over to smooth test functions. Another consequence is that $z_1(x,t) \le z_2(x,t)$ if and only if $u(x,t) \le v(x,t)$. Hence Theorem 4.1.4 follows from proving the same comparison result for sub- and supersolutions of (4.1.6).

We proceed with the usual scheme of doubling variables. Assume that the conclusion of the theorem is false and $M := \sup_{\mathbb{R}^N \times (0,\infty)} z_1 - z_2 > 0$. Define first, for $\delta > 0$,

$$M_\delta := \sup_{\mathbb{R}^N \times (0,\infty)} z_1(x) - z_2(x) - \delta(|x|^2 + t). \tag{4.1.8}$$

As the penalized function on the right-hand side of (4.1.8) goes to $-\infty$ as $|x| \to \infty$, the supremum in achieved at some $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0,\infty)$. From standard arguments, we have that $M_\delta \to 0$ as $\delta \to 0$ (see e.g., [25], Proposition 3.7), which implies that for small enough δ, $M_\delta > 0$. Since z_1 is nonpositive in all of $\mathbb{R}^N \times (0,\infty)$, this implies $z_2(\bar{x}, \bar{t}) > 0$, i.e. $v(\bar{x}, \bar{t}) < +\infty$. Moreover, since $z_1(\cdot, 0) \le z_2(\cdot, 0)$, we necessarily have $\bar{t} > 0$. Define now

$$M_{\delta,\alpha} := \sup_{\mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)} z_1(x,t) - z_2(y,t) - \frac{\delta}{2}(|x|^2 + |y|^2 + t) - \frac{\alpha}{2}|x-y|^2.$$

Again the supremum above if achieved at a point $(\hat{x}, \hat{y}, \hat{t})$. Similarly, we know that as $\alpha \to \infty$ and δ remains fixed, $M_{\alpha,\delta} \to M_\delta$ and $\hat{x}, \hat{y} \to \bar{x}$ and $t \to \bar{t}$ for $\bar{x}, \bar{t}$ as above (again, up to a subsequence; note also that such $\bar{x}, \bar{t}$ are not necessarily unique).

Thus, an application of Ishii's Lemma (see [25], Theorem 3.2) gives

$$N(\hat{x}, z_1(\hat{x}, \hat{t}), \alpha(\hat{x}-\hat{y}) + \delta\hat{x}) - N(\hat{y}, z_2(\hat{y}, \hat{t}), \alpha(\hat{x}-\hat{y}) - \delta\hat{y}) \le \left(2n + \frac{1}{2}\right)\delta. \tag{4.1.9}$$

We aim to bound this difference from below by a positive constant independent of δ. Let

$$h(s) = N(a(s), b(s), c(s)),$$

where

$$\begin{aligned} &a(s) = s\hat{x} + (1-s)\hat{y}, \qquad b(s) = sz_1(\hat{x}) + (1-s)z_2(\hat{y}), \\ &c(s) = \alpha(\hat{x}-\hat{y}) + \delta(s\hat{x} + (s-1)\hat{y}). \end{aligned}$$

We rewrite (4.1.9) as

$$\begin{aligned} &N(\hat{x}, z_1(\hat{x}), \alpha(\hat{x}-\hat{y}) + \delta\hat{x}) - N(\hat{y}, z_2(\hat{y}), \alpha(\hat{x}-\hat{y}) - \delta\hat{y}) \\ &= N(a(1), b(1), c(1)) - N(a(0), b(0), c(0)) \\ &= h(1) - h(0) = \int_0^1 h'(s)\, ds. \end{aligned} \tag{4.1.10}$$

Computing

$$a'(s) = \hat{x} - \hat{y}, \qquad b'(s) = z_1(\hat{x}) - z_2(\hat{y}),$$
$$c'(s) = \delta(\hat{x} + \hat{y}),$$

and, from (4.1.7),

$$\frac{\partial N}{\partial x} = r\, Df(x), \qquad \frac{\partial N}{\partial r} = f(x) - \left|\frac{p}{r}\right|^2 + (m-1)\left|\frac{p}{r}\right|^m,$$
$$\frac{\partial N}{\partial p} = \left(\frac{2}{|r|^2} - \frac{m(m-2)|p|^{m-2}}{|r|^m}\right) p,$$

we have

$$\int_0^1 h'(s)\, ds = \int_0^1 \frac{\partial N}{\partial x}(a(s), b(s), c(s)) \cdot a'(s) + \frac{\partial N}{\partial r}(a(s), b(s), c(s))b'(s) + \frac{\partial N}{\partial p}(a(s), b(s), c(s)) \cdot c'(s)\, ds.$$

We proceed to bound each term in the last expression:

$$\frac{\partial N}{\partial x}(a(s), b(s), c(s)) \cdot a'(s) \geq -\max\{\|z_1\|_\infty, \|z_2\|_\infty\}\|Df\|_{\infty, B_{R(\delta)}}|\hat{x} - \hat{y}|,$$

where $B_{R(\delta)}$ is a ball that contains $(\hat{x}, \hat{y})$, the points at which the maximum of $\Phi = \Phi^{\delta,\alpha}(x, y)$ is achieved, considering $\delta > 0$ is fixed; $B_{R(\delta)}$ is uniform in α. Since $\hat{x} - \hat{y} \to 0$ as $\alpha \to \infty$, this implies

$$\frac{\partial N}{\partial x}(a(s), b(s), c(s)) \cdot a'(s) \to 0 \quad \text{as } \alpha \to \infty.$$

We write $q := \frac{p}{r}$. Young's inequality gives

$$|q|^2 \leq \frac{2}{m}|q|^m + \frac{m}{m-2},$$

hence

$$\begin{aligned}\frac{\partial N}{\partial r} &= f(x) - \left|\frac{p}{r}\right|^2 + (m-1)\left|\frac{p}{r}\right|^m \\ &\geq f(x) + (m-1)|q|^m - \frac{2}{m}|q|^m - \frac{m}{m-2} \\ &\geq f(x) - \frac{m}{m-2} + \left(m - 1 - \frac{2}{m}\right)|q|^m \\ &\geq 1 + C|q|^m, \end{aligned} \tag{4.1.11}$$

taking $0 < C < m - 1 - \frac{2}{m}$ in the last inequality. The bound

$$f(x) \geq 1 + \frac{m}{m-2} \quad \text{for all } x \in \mathbb{R}^N, \tag{4.1.12}$$

may be assumed without loss of generality by initially considering, instead of u and v, the functions

$$u(x,t) + \left(1 + \frac{m}{m-2}\right) t, \qquad v(x,t) + \left(1 + \frac{m}{m-2}\right) t,$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. On the other hand, by direct computation,

$$\begin{aligned} \left|\frac{\partial N}{\partial p}\right| &= 2\left|\frac{p}{r}\right| + m\left|\frac{p}{r}\right|^{m-1} = 2|q| + m|q|^{m-1} \\ &\leq m(|q| + |q|^{m-1}) \leq 2m(1 + |q|^m). \end{aligned}$$

Here we've used that $m > 2$ and that $|q| + |q|^{m-1} \leq 2(1 + |q|^m)$, which follows easily by considering the cases $|q| > 1$ and $|q| \leq 1$ separately. We have therefore obtained

$$\left(1 + \frac{2m}{C}\right) \frac{\partial N}{\partial r} \geq \left|\frac{\partial N}{\partial p}\right|. \tag{4.1.13}$$

Thus, using (4.1.11) and (4.1.13), for small $\delta > 0$ we have

$$\begin{aligned} \liminf_{\alpha\to\infty} \int_0^1 h'(s)\,ds &\geq \liminf_{\alpha\to\infty} \int_0^1 \frac{\partial N}{\partial r}(z_1(\hat{x}) - z_2(\hat{y})) + \frac{\partial N}{\partial p} \cdot \delta(\hat{x} + \hat{y})\,ds \\ &\geq \liminf_{\alpha\to\infty} \int_0^1 \frac{\partial N}{\partial r}\left(M_{\alpha,\delta} + \delta(|\hat{x}|^2 + |\hat{y}|^2 + \hat{t}) + \frac{\alpha}{2}|\hat{x} - \hat{y}|^2\right) \\ &\quad + \frac{\partial N}{\partial p} \cdot \delta(\hat{x} + \hat{y})\,ds \\ &\geq \liminf_{\alpha\to\infty} \int_0^1 \frac{\partial N}{\partial r} M_{\alpha,\delta} - \delta\left|\frac{\partial N}{\partial p}\right|(|\hat{x}| + |\hat{y}|)\,ds \\ &\geq \int_0^1 \frac{\partial N}{\partial r}\left(M_\delta - 2\delta\left(1 + \frac{2m}{C}\right)|\bar{x}|\right) ds \\ &> \frac{M}{2} > 0. \end{aligned} \tag{4.1.14}$$

Here we have also used the facts concerning the limits $\alpha \to \infty$, $\delta \to 0$ (taken in that order) mentioned at the outset of the argument. Together with (4.1.9) and (4.1.10), (4.1.14) gives the desired contradiction. □

Remark 4.1.5. We note that Theorem 4.1.4 does not require f to be nonnegative, as does Theorem 1.2.6, but merely bounded from below. Moreover, assuming the bound (4.1.12) replaces the assumption of the coercivity of f, which is used in the analogue of Theorem 4.1.4 for the ergodic problem in [15].

4.2 Sub- and supersolutions

In this section we construct sub- and supersolutions to a modified evolution problem. This will allow us to relate the solution of (1.2.15)-(1.2.16) to the solution of the ergodic problem,

$$\lambda - \Delta\phi + |D\phi|^m = f(x) \quad \text{in } \mathbb{R}^N, \tag{4.2.1}$$

where both λ and ϕ are unknown. We now state important properties of the solution ϕ which depend on the growth rate for f given by assumptions (H1) and (H2), stated in the introduction. Assuming only (H1), it is shown in [15], Proposition 3.3, that there exists $K > 0$ such that, for all $x \in \mathbb{R}^N$,

$$|D\phi(x)| \leq K(1 + |x|^{\gamma-1}) \quad \text{and} \quad |\phi(x)| \leq K(1 + |x|^{\gamma}), \tag{4.2.2}$$

where $\gamma = \frac{\alpha}{m} + 1$. On the other hand, (H1)-(H2) together imply that ϕ indeed has polynomial growth at infinity: it is shown in [15], Proposition 3.4, that there exists $c > 0$ such that

$$\phi(x) \geq c|x|^{\gamma} - c^{-1} \quad \text{for all } x \in \mathbb{R}^N, \tag{4.2.3}$$

with γ as above. In particular, this implies that ϕ is coercive.

Lemma 4.2.1. *Under the assumptions* (H1) *and* (H2)*, there exist* $U \in USC(\mathbb{R}^N \times (0, \infty))$ *and* $V \in LSC(\mathbb{R}^N \times (0, \infty); \mathbb{R} \cup \{+\infty\})$, $\mathrm{dom}(V) \neq \emptyset$, *which are respectively a sub- and a supersolution of*

$$u_t - \Delta u + |Du|^m = f(x) - \lambda^* \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.2.4}$$

Furthermore, both U and V are bounded from below, and satisfy the following:

(i) There exists a constant $\sigma > 0$ such that $U(\cdot, t) \to \phi - \sigma$ and $V(\cdot, t) \to \phi + \sigma$ locally uniformly as $t \to \infty$.

(ii) For any fixed $\hat{t} > 0$, there exists $R > 0$ such that

$$V(x, \hat{t}) = +\infty \quad \text{for all } |x| \geq R.$$

(iii) There exists $M > 0$ such that, for all $t > 0$,

$$U(x, t) \leq t + M \quad \text{for all } x \in \mathbb{R}^N.$$

Proof. We begin with the construction of the supersolution V, and will later indicate the necessary significant changes to obtain U. While the constructions are similar, it is not the case where one can be obtained from the other. On first approach, define

$$v(x, t) = \phi(x) + \chi(\phi(x) - R(t)), \tag{4.2.5}$$

where $R : \mathbb{R} \to \mathbb{R}$, $\chi : I \subset \mathbb{R} \to \mathbb{R}$ are smooth functions to be chosen, as is the interval $I := (0, b)$, $b > 0$. Thus, v is smooth wherever it is defined. Using

(4.2.1), we compute

$$\begin{aligned} &v_t - \Delta v + |Dv|^m - f(x) + \lambda^* \\ &\quad = -\chi' \dot{R}(t) - (1+\chi')(\lambda^* - f(x) + |D\phi|^m) - \chi''|D\phi|^2 \\ &\qquad + (1+\chi')^m |D\phi|^m - f(x) + \lambda^* \\ &\quad = \chi'(-\dot{R}(t) + f(x) - \lambda^*) + ((1+\chi')^m - (1+\chi'))|D\phi|^m - \chi''|D\phi|^2 \\ &\quad \geq \chi'(-\dot{R}(t) + f(x) - \lambda^*) + (1 - 2/m)((1+\chi')^m - (1+\chi'))|D\phi|^m \\ &\qquad - \frac{m-2}{m}\left(\frac{\chi''}{((1+\chi')^m - (1+\chi'))^{\frac{2}{m}}}\right)^{\frac{m}{m-2}} \end{aligned} \tag{4.2.6}$$

where we've used Young's inequality to control the term with $|D\phi|^2$. In χ and its derivatives the argument $\phi(x) - R(t)$ is omitted to ease notation.

Motivated by (4.2.6), we define $\chi(s) = 0$ for $s \leq 0$, and for $s > 0$ take χ to be the solution of the ODE

$$\begin{aligned} &\chi'' = C(\chi')^{\beta_1}(1+\chi')^{\beta_2} \quad \text{in } (0, b), \\ &\chi(0) = \chi'(0) = 0, \end{aligned} \tag{4.2.7}$$

with $\beta_1, \beta_2 > 0$ to be chosen, and $b > 0$ determined by β_1 and β_2. We note that (4.2.7) can be seen as a first-order ODE. Taking $\beta_1 < 1$ avoids the trivial solution $\chi \equiv 0$, since in this case (4.2.7) fails to meet Osgood's condition (see e.g., [1]). Furthermore, if β_2 is chosen so that $\beta_1 + \beta_2 > 1$, we have that $\chi(s) \to +\infty$ as $s \to b^-$ for $b > 0$ as above. It can also be shown that

$$\chi'(s), \chi''(s) > 0 \quad \text{for all } s > 0. \tag{4.2.8}$$

In particular, this justifies the use of Young's inequality in (4.2.6).

We extend the definition of χ as follows:

$$\tilde{\chi}(s) = \begin{cases} \chi(s), & \text{if } s \in (-\infty, b), \\ +\infty, & \text{if } s \geq b. \end{cases} \tag{4.2.9}$$

Thus, defining

$$\tilde{v}(x,t) = \phi(x) + \tilde{\chi}(\phi(x) - R(t)),$$

we have that $\tilde{v} \in LSC(\mathbb{R}^N \times (0,\infty); \mathbb{R} \cup \{+\infty\})$, and $\tilde{v}$ is smooth in $\mathrm{dom}(\tilde{v})$. In what follows we drop the tilde from $\tilde{v}$ and $\tilde{\chi}$ for simplicity.

Coming back to (4.2.6), specifically, to the last expression obtained before applying Young's inequality, we see that given our choice of χ, if $\phi(x) - R(t) \leq 0$ then immediately

$$v_t - \Delta v + |Dv|^m - f(x) + \lambda^* \geq 0.$$

Assume now that $\phi(x) - R(t) > 0$, but is sufficiently small. This implies that χ' is also small. Under this assumption, we aim to control the last term in (4.2.6) by a positive power of χ', i.e., to obtain for some $C > 0$ and $0 < \beta < 1$,

$$\left(\frac{C(\chi')^{\beta_1}(1+\chi')^{\beta_2}}{((1+\chi')^m - (1+\chi'))^{\frac{2}{m}}}\right)^{\frac{m}{m-2}} \leq (\chi')^{\beta}. \tag{4.2.10}$$

Furthermore, we aim to have β arbitrarily close to 1. As χ' is small, we compute

$$\begin{aligned}(1+\chi')^m - (1+\chi') &= 1 + m\chi' + \sum_{k\geq 2}\binom{m}{k}(\chi')^k - 1 - \chi' \\ &= (m-1)\chi' + o(1) \quad (\text{as } \chi' \to 0) \\ &\geq (m-2)\chi'. \end{aligned} \tag{4.2.11}$$

Since $(1+\chi')^{\beta_2 \frac{m}{m-2}} \to 1$ for $\chi' \to 0$, we control this term with the constant C on the left of (4.2.10). Using (4.2.11) and taking small enough $C > 0$ we have

$$\begin{aligned}\left(\frac{C(\chi')^{\beta_1}(1+\chi')^{\beta_2}}{((1+\chi')^m - (1+\chi'))^{\frac{2}{m}}}\right)^{\frac{m}{m-2}} &\leq \left(C\frac{(\chi')^{\beta_1}}{(\chi')^{\frac{2}{m}}}\right)^{\frac{m}{m-2}} \\ &\leq (\chi')^{\frac{\beta_1 m-2}{m-2}}.\end{aligned}$$

This gives (4.2.10) by taking β_1 sufficiently close to 1. Plugging this into (4.2.6), noting the term in $|D\phi|^m$ is nonnegative, we now have

$$\begin{aligned}&v_t - \Delta v + |Dv|^m - f(x) + \lambda^* \\ &\quad \geq \chi'(-\dot{R}(t) + f(x) - \lambda^*) - (\chi')^\beta. \end{aligned} \tag{4.2.12}$$

Recall that we are working under the assumption that $\phi(x) > R(t)$. Set $R(t) = t + t_0$, for some $t_0 > 0$ to be chosen. From (4.2.2) we then have,

$$t + t_0 < \phi(x) \leq K(1+|x|^\gamma).$$

Combining this with (H2), we obtain

$$f_0^{-1}\left(\left[\left(\frac{t+t_0-K}{K}\right)^+\right]^{\frac{\alpha}{\gamma}} + 1\right) \leq f(x). \tag{4.2.13}$$

Thus, taking t_0 sufficiently large, we have

$$f(x) - \lambda^* - 1 > 0. \tag{4.2.14}$$

From our choice of $R = R(t)$, we now have $\dot{R}(t) = 1$. We use (4.2.14) and Young's inequality to estimate

$$\begin{aligned}&v_t - \Delta v + |Dv|^m - f(x) + \lambda^* \\ &\quad \geq (1-\beta)\left(\chi'(f(x) - \lambda^* + 1) - (f(x) - \lambda^* + 1)^{-\frac{\beta}{1-\beta}}\right) \\ &\quad \geq -(1-\beta)(f(x) - \lambda^* + 1)^{-\frac{\beta}{1-\beta}}. \end{aligned} \tag{4.2.15}$$

Further still, (4.2.13) implies, after a simple computation, that for some $0 < \hat{\alpha} < \frac{\alpha}{\gamma}$ and large enough t_0, we have

$$t^{\hat{\alpha}} + 1 \leq f(x) - \lambda^* - 1,$$

and consequently,

$$(f(x) - \lambda^* - 1)^{-\frac{\beta}{1-\beta}} \leq (t^{\hat{\alpha}} + 1)^{-\frac{\beta}{1-\beta}}, \quad \text{for all } t > 0. \tag{4.2.16}$$

We write $\hat{\beta} = \frac{\beta}{1-\beta}$, and let

$$\psi(t) = (1 - \beta) \int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}} \, d\tau.$$

Since $\hat{\beta}$ can be made arbitrarily large by taking β close to 1 and $\hat{\alpha} > 0$, the integral defining ψ remains bounded as $t \to \infty$. We write

$$\sigma := (1 - \beta) \int_0^\infty (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}} \, d\tau < \infty. \tag{4.2.17}$$

Note also, ψ is smooth and $\psi'(t) = (1 - \beta)(t^{\hat{\alpha}} + 1)^{-\hat{\beta}} > 0$ for all $t > 0$. Thus, to recap, defining

$$V(x, t) = v(x, t) + \psi(t), \tag{4.2.18}$$

with v and ψ as above, yields, by (4.2.15) and (4.2.16), that

$$\partial_t V - \Delta V + |DV|^m - f(x) + \lambda^* \geq 0$$

for all (x, t) such that $\phi(x) - (t + t_0) < \delta$, for some small $\delta > 0$. In particular, $\delta < b$, where b is given by (4.2.7). We also note that V is bounded from below since ϕ is bounded from below and $\chi, \psi \geq 0$, and $V \in C^{2,1}(\text{dom}(V))$ since $\phi \in C^2(\mathbb{R}^N)$ and χ and ψ are both smooth.

We proceed to check that V is a supersolution of (4.2.4) for the remaining values of (x, t). Assume now that $s := \phi(x) - (t + t_0) \in [\delta, b - \delta]$. This implies that $\chi'(s) > \bar{\delta}$ for some $\bar{\delta} > 0$. In this case, the expressions on either side of (4.2.10) are continuous, hence remain bounded, for $s \in [\delta, b - \delta]$. Thus (4.2.10) is obtained by choosing an appropriately small C. Finally, we address the case $s > b - \delta$. Since $\delta > 0$ is small and $\chi'(s) \to \infty$ as $s \to b$, this amounts to checking (4.2.10) in the the limit $\chi' \to \infty$. Using only that $m > 2$, a straightforward computation then shows that setting $1 < \beta_2 < 2$ in (4.2.7), the left-hand side of (4.2.10) vanishes as $\chi' \to \infty$, while the right-hand side goes to infinity. Thus V is a supersolution on all of $\mathbb{R}^N \times (0, \infty)$.

We move on to proving V satisfies (i) and (ii). Let $K \subset \mathbb{R}^N$ be a compact set. Then, for large enough $t > 0$, so that $t > \phi(x) - t_0$ for all $x \in K$, we have $\chi(\phi(x) - (t + t_0)) = 0$, by the definition of χ. Thus, as $t \to \infty$,

$$V(x, t) = \phi(x) + (1 - \beta) \int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}} \, d\tau \to \phi(x) + \sigma,$$

uniformly over K. On the other hand, for fixed $\hat{t} > 0$, by (4.2.3), we take $R > 0$ large enough so that $\phi(x) > b + \hat{t} + t_0$ for all $|x| \geq R$. This implies that $\chi(\phi(x) - (\hat{t} + t_0)) = +\infty$, hence $V(x, \hat{t}) = +\infty$, for all $|x| \geq R$.

We proceed now to constructing the subsolution U of (4.2.4). As mentioned, the construction is similar, so we merely go over the main points. Define

$$u(x,t) = t + t_0 + \xi(\phi(x) - (t + t_0))$$

where again, $t_0 > 0$ and $\xi \in C^\infty(\mathbb{R}^N)$ are to be chosen. We set

$$\xi(s) = s \tag{4.2.19}$$

for $s \leq 0$, and for $s > 0$, define ξ as the solution of

$$\xi'' = -C(1-\xi)^{\eta_1}(\xi')^{\eta_2} \quad \text{in } (0,\infty), \qquad \xi(0) = 0,\ \xi'(0) = 1, \tag{4.2.20}$$

with $0 < \eta_1 < 1$, $\eta_2 > 0$ to be chosen. It can be shown that

$$\xi(s) > 0,\ -C \leq \xi''(s) < 0 \quad \text{and} \quad 0 < \xi'(s) < 1, \qquad \text{for all } s > 0, \tag{4.2.21}$$

for some $C > 0$. By taking $\eta_1 + \eta_2 > 1$, we also have that

$$\xi(s) \uparrow M \text{ as } s \to \infty, \tag{4.2.22}$$

for some $M > 0$ depending on η_1 and η_2.

Computing as in (4.2.6), it is seen that u is a subsolution trivially for $\phi(x) - (t + t_0) \leq 0$, whereas, if $\phi(x) - (t + t_0) > 0$, we have

$$\begin{aligned} &u_t - \Delta u + |Du|^m - f(x) + \lambda^* \\ &\quad \leq -(1 - \xi')(f(x) - \lambda^* - 1) + \frac{m-2}{m}\left(\frac{-\xi''}{[\xi' - (\xi')^m]^{\frac{2}{m}}}\right)^{\frac{m}{m-2}}. \end{aligned}$$

Assuming then that $0 < \phi(x) - (t + t_0) < \delta$ for some small $\delta > 0$, we use (4.2.20) and the fact that $(\xi')^{m-1} \geq (m-2)(1-\xi)$ for small ξ' (shown again by a series expansion), to obtain

$$\left(\frac{-\xi''}{[\xi' - (\xi')^m]^{\frac{2}{m}}}\right)^{\frac{m}{m-2}} \leq (1 - \xi')^{\frac{\eta_1 m - 2}{m-2}}.$$

Thus, choosing η_1 close to 1 we obtain

$$u_t - \Delta u + |Du|^m - f(x) + \lambda^* \leq -(1 - \xi')(f(x) - \lambda^* - 1) + (1 - \xi')^\beta,$$

with $0 < \beta < 1$, β arbitrarily close to 1 (we can take the same value as before). From here on we argue we did earlier to conclude that

$$U(x,t) := u(x,t) - (1 - \beta)\int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau$$

is a subsolution in all of $\mathbb{R}^N \times (0,\infty)$. Using (4.2.19), we prove U satisfies (i) as we did before for V. Finally, (4.2.22) and (4.2.21) imply (iii), by noting that $\xi(s) \leq s$ for all $s \in \mathbb{R}$. □

4.3 Large-time behaviour

In this final section, let $u = u(x,t)$ denote the unique solution of (1.2.15)-(1.2.16) given by Theorem 1.2.6. We also use the notation from Section 4.2.

Lemma 4.3.1. *Under assumptions* (H1) *and* (H2)*,* $u(x,t) - \lambda^* t$ *is bounded over compact sets, uniformly with respect to* $t > 0$*.*

Proof. Let $K \subset \mathbb{R}^N$ be compact. We can take K sufficiently large, i.e. $B_R \subset K$ for large $R > 0$, with no loss of generality, since uniform boundedness on K implies uniform boundedness on any subset. The proof follows by comparing $u(x,t)-\lambda^* t$ to the sub- and supersolutions of Lemma 4.2.1, noting that $u(x,t)-\lambda^* t$ is a solution of (4.2.4).

Fix $\hat{x} \in K$. Recalling the construction of the supersolution V, we take t_0 large enough (if necessary) so that $\phi(\hat{x})-t_0 < b$. This implies that $V(\hat{x},0) < \infty$, and is thus proper. Furthermore, using Lemma 4.2.1 (ii), we take $R > 0$ (where $B_R \subset K$) large enough so that $\phi(x) - t_0 > b$ for all $|x| \geq R$. This implies that $V(x,0) = +\infty$ for all $x \in \mathbb{R}^N \backslash B_R \supset \overline{\mathbb{R}^N \backslash K}$.

Hence,

$$V(x,0) + \sup_K u_0 \geq u_0(x) \quad \text{for all } x \in \mathbb{R}^N.$$

Thus, $V(x,t)+\sup_K u_0$ is a supersolution of (4.2.4)-(1.2.16), and by comparison (Theorem 4.1.4),

$$V(x,t) + \sup_K u_0 \geq u(x,t) - \lambda^* t \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{4.3.1}$$

Furthermore, by Lemma 4.2.1 (i),

$$V(x,t) + \sup_K u_0 \to \phi(x) + \sigma + \sup_K u_0, \quad \text{uniformly over } K \text{ as } t \to \infty.$$

Therefore, taking $C > \sup_{K\times[0,T]} u(x,t) - \lambda^* t$ for some large $T > 0$ given by the previous limit, we have

$$u(x,t) - \lambda^* t \leq \sup_K \phi + \sigma + \sup_K u_0 + C \quad \text{for all } x \in K,\ t > 0.$$

Similarly, we use the subsolution U from Lemma 4.2.1 to obtain the lower bound. Recalling u_0 is assumed nonnegative, by Lemma 4.2.1 (iii), we have

$$U(x,0) - M \leq 0 \leq u_0(x) \quad \text{for all } x \in \mathbb{R}^N,$$

hence, by comparison,

$$U(x,t) - M \leq u(x,t) - \lambda^* t \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Thus for large $t > 0$, by Lemma 4.2.1 (i),

$$\phi(x) - \sigma - M \leq u(x,t) - \lambda^* t \quad \text{for all } x \in K,$$

and finally,

$$\inf_K \phi(x) - \sigma - M - C \leq u(x,t) - \lambda^* t \quad \text{for all } x \in K, t > 0,$$

by taking $C > 0$ as before. □

Remark 4.3.2. An immediate consequence of Lemma 4.3.1 is the preliminary result that

$$\frac{u(x,t)}{t} \to \lambda \quad \text{locally uniformly in } \mathbb{R}^N \text{ as } t \to \infty.$$

We proceed to prove the convergence result Theorem 1.2.7, stated in the introduction.

Proof of Theorem 1.2.7. The assumption that $u_0 \in C^2(\mathbb{R}^N)$ implies, by Proposition 4.1 in [60], that, for any compact $K \subset \mathbb{R}^N$, $u(\cdot,t) \in C^{\frac{m-2}{m-1}}(K)$ for all $t > 0$, and moreover, that $\|u(\cdot,t)\| \leq C$ for some $C > 0$ independent of t. In short, Proposition 4.1 in [60] is obtained by showing that u is Lipschitz continuous with respect to t, with Lipschitz constant depending on $\|u_0\|_{C^2(K)}$, applying the corresponding result for stationary equations from [23]. These estimates are crucial to the compactness argument of Step 2, below, which is slightly modified version of the one used in [60].

Step 1. For simplicity we write $v(x,t) := u(x,t) - \lambda^* t$. Lemma 4.3.1 implies that

$$\bar{v}(x) = \limsup_{t\to\infty}{}^* v(x,t) < +\infty \quad \text{for all } x \in \mathbb{R}^N.$$

Therefore, by the stability of viscosity solutions, $\bar{v}$ so defined is a subsolution of (4.2.1) in all of $\mathbb{R}^N$. Hence, adding an appropriate constant to either ϕ of $\bar{v}$, by the strong maximum principle we have that

$$\bar{v}(x) = \phi(x) + \hat{c} \quad \text{for all } x \in \mathbb{R}^N,$$

for some $\hat{c} \in \mathbb{R}^N$ (See the proof of Theorem 3.1 in [15] for details).

Step 2. For fixed $\hat{x} \in \mathbb{R}^N$, by the definition of half-relaxed limits, there exists a sequence $(x_n, t_n) \in \mathbb{R}^N \times (0,\infty)$ such that $x_n \to \hat{x}$, $t_n \to \infty$, and $v(x_n,t_n) \to \bar{v}(\hat{x})$. Consider $v_n(\cdot) := v(\cdot, t_n - 1)$. By Lemma 4.3.1, the sequence (v_n) is uniformly bounded over compact sets. Furthermore, by the Hölder estimate from Proposition 4.1 in [60], it is also locally uniformly equicontinuous. Thus, there exists a subsequence $(v_{n'})$ such that $v_{n'} \to w_0$ locally uniformly in $\mathbb{R}^N$ as $n' \to \infty$, for some $w_0 \in C(K)$.

Define now

$$w_{n'}(x,t) = v(x, t + t_{n'} - 1) \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,\infty),$$

Again, by compactness, recalling that v is Lipschitz continuous with respect to t, and hence, so is each $w_{n'}$, we have that $w_{n'}(\cdot,t) \to w(\cdot,t)$ locally uniformly in $\mathbb{R}^N \times (0,T)$, for some $w \in C(\mathbb{R}^N \times \infty)$ (passing to a subsequence yet again, if necessary; we omit this for ease of notation). By stability of viscosity solutions, w is a solution of (4.2.4) in $\mathbb{R}^N \times (0,T)$. Note also that $w(x,0) = \lim v(x, t'_n - 1) = w_0(x)$ for all $x \in \mathbb{R}^N$.

By the definition of the half-relaxed limit $\bar{v}$, $w(x,t) \le \bar{v}(x)$ for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$, and by construction,

$$w(\hat{x},1) = \lim_{n'} v(\hat{x}, t_{n'}) = \phi(\hat{x}) + \hat{c}. \tag{4.3.2}$$

Hence, by the parabolic maximum principle (see, e.g., Lemma 2.1 in [60]), w is constant in $\mathbb{R}^N \times [0,1]$. Thus $w_0(x) = \phi(x) + \hat{c}$ for all $x \in \mathbb{R}^N$.

Thus, to summarize this Step, we have obtained that, given any compact $K \subset \mathbb{R}^N$, the exists a sequence $t_{n'} - 1 \to \infty$ such that

$$v(\cdot, t_{n'} - 1) \to \phi + \hat{c} \quad \text{uniformly over } K \text{ as } t_{n'} - 1 \to \infty. \tag{4.3.3}$$

In the next Step, we will use (4.3.3) for $K = \overline{B}_R$, the closed ball of radius R, for a suitably chosen large $R > 0$.

Step 3. Let $\epsilon > 0$ and $\widehat{K} \subset \mathbb{R}^N$ be compact (this is the set on which we will prove the uniform convergence stated in the Theorem). For this final part of the proof we employ many of the elements and arguments of Lemma 4.2.1. We refer the reader to that Lemma, in particular for notation.

For $R > 0$ and $(x,t) \in \mathbb{R}^N \times (0,\infty)$, we define

$$V_R(x,t) = \phi(x) + \hat{c} + \chi(\phi(x) + \hat{c} - (t+R)) + \int_R^{t+R} (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau + \frac{1}{R}, \tag{4.3.4}$$

with $\chi, \hat{\alpha}$ and $\hat{\beta}$ are as in the definition of the supersolution V. Thus, arguing as in the proof of Lemma 4.2.1, we have that for sufficiently large $R > 0$, V_R is a supersolution of (4.2.4).

Recall that in the growth estimate (4.2.3), $\gamma = \frac{\alpha}{m} + 1 > 1$. Hence, we may take $R > 0$ large enough so that $\widehat{K} \subset B_R$ and

$$c^{-1}R^{\gamma} - c + \hat{c} - R > b, \tag{4.3.5}$$

with b as in (4.2.9), i.e., such that $\chi(s) \equiv +\infty$ for $s \ge b$. Thus, by (4.3.5) and (4.2.9), we have that

$$V_R(x,0) \equiv +\infty \quad \text{for all } x \in \mathbb{R}^N \backslash B_R. \tag{4.3.6}$$

We now use (4.3.3) from Step 2 for $K = \overline{B}_R$ to obtain that, for large enough n',

$$v(x, t_{n'} - 1) < \phi(x) + \hat{c} + \frac{1}{R} \quad \text{for all } x \in \overline{B}_R.$$

This gives that $v(x, t_{n'} - 1) \le V_R(x,0)$ in $\overline{B}_R$, by construction. Together with (4.3.6), this implies

$$v(x, t_{n'} - 1) \le V_R(x,0) \quad \text{for all } x \in \mathbb{R}^N. \tag{4.3.7}$$

Since v and V_R are a solution and a supersolution of (4.2.4), respectively, (4.3.7) and Theorem 4.1.4 imply

$$v(x, t + t_{n'} - 1) \leq V_R(x,t) \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.3.8}$$

We note that $\int_R^\infty (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau + \frac{1}{R} = o(1)$ as $R \to \infty$. Thus, arguing as in the proof of Lemma 4.2.1 (i), we take $R > 0$ larger still so that,

$$\begin{aligned} \widehat{V}_R(x) &:= \phi(x) + \hat{c} + \chi(\phi(x) + \hat{c} - R) + \int_R^\infty (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau + \frac{1}{R} \\ &< \phi(x) + \hat{c} + \epsilon \quad \text{for all } x \in \widehat{K}. \end{aligned} \tag{4.3.9}$$

By (4.2.8), $t \mapsto \chi(\phi(x) + \hat{c} - (t + R))$ is nonincreasing, hence

$$V_R(x,t) \leq \widehat{V}_R(x) \quad \text{ for all } t > 0,\ \hat{x} \in \widehat{K}. \tag{4.3.10}$$

Therefore, combining (4.3.8), (4.3.9) and (4.3.10), we obtain

$$v(x, t + t_{n'} - 1) \leq \phi(x) + \hat{c} + \epsilon \quad \text{for all } x \in \widehat{K},\ t > 0. \tag{4.3.11}$$

To obtain the lower bound corresponding to (4.3.11), we define the analogue of V_R based on the subsolution U from Lemma 4.2.1. For $(x,t) \in \mathbb{R}^N \times (0, \infty)$, define

$$U_R(x,t) = t + R + \xi(\phi(x) + \hat{c} - (t + R)) - \int_R^{t+R} (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau - \frac{1}{R},$$

with $\xi, \hat{\alpha}$ and $\hat{\beta}$ as before. Consider now

$$\underline{v}(x) = \liminf_{t \to \infty} {}_* \, v(x,t).$$

Arguing as in Step 1, it follows that

$$\underline{v}(x) = \phi(x) + m^-,$$

for some $m^- \in \mathbb{R}$. By definition of the half-relaxed limit, we have

$$\phi(x) + m^- \leq v(x, t_{n'} - 1) \quad \text{for all } x \in \mathbb{R}. \tag{4.3.12}$$

for sufficiently large n'. Recall from the construction of Lemma 4.2.1 that $\xi(s) \leq M$ for all $s \in \mathbb{R}$, for some $M \in \mathbb{R}$. We set $R > 0$ large enough so that U_R is a subsolution of (4.2.4) and

$$c^{-1} R^\gamma - c - R \geq M - m^-. \tag{4.3.13}$$

We remark that this choice is independent of all the previous requirements made for $R > 0$ in the construction of V_R. We then have, by (4.2.3), (4.3.12) and (4.3.13),

$$\begin{aligned} U_R(x,0) &\leq R + \xi(\phi(x) + \hat{c} - R) \leq R + M \leq c^{-1} R^\gamma - c + m^- \\ &\leq \phi(x) + m^- \leq v(x, t_{n'} - 1) \quad \text{for all } x \in \mathbb{R}^N \backslash B_R. \end{aligned}$$

By (4.3.3) we have, for sufficienly large n',

$$v(x, t_{n'} - 1) > \phi(x) + \hat{c} - \frac{1}{R} \quad \text{for all } x \in \overline{B}_R$$

Recalling that $\xi(s) \leq s$ for all $s \in \mathbb{R}$, we have

$$U_R(x,0) \leq R + \xi(\phi(x) + \hat{c} - R) - \frac{1}{R} \leq \phi(x) + \hat{c} - \frac{1}{R} \quad \text{for all } x \in \mathbb{R}^N.$$

Thus we obtain

$$U_R(x,0) \leq v(x, t_n - 1) \quad \text{for all } x \in \mathbb{R}^N. \tag{4.3.14}$$

Together with Theorem 4.1.4, this implies that

$$U_R(x,t) \leq v(x, t + t_n - 1) \quad \text{for all } x \in \mathbb{R}^N,\ t > 0. \tag{4.3.15}$$

From this point on, we argue as we did before for V_R. We remark that the analogue of (4.3.10) (for a similarly defined $\widehat{U}_R(x)$) is now given by the fact that the function $t \mapsto t + R + \xi(\phi(x) + \hat{c} - (t + R))$ is nondecreasing, since $0 \leq \xi'(s) \leq 1$ for all $s \in \mathbb{R}$, by (4.2.21). Thus, for large enough n', depending on $R > 0$, we have

$$v(x, t + t_{n'} - 1) \geq \phi(x) + \hat{c} - \epsilon, \quad \text{for all } x \in \widehat{K},\ t > 0,$$

and with this we conclude. □

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